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ARTICLE I.

A NEW METHOD OF DETERMINING THE GENERAL PERTURBATIONS OF THE MINOR PLANETS.

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PREFACE.

In determining the general perturbations of the minor planets the principal difficulty arises from the large eccentricities and inclinations of these bodies. Methods that are applicable to the major planets fail when applied to the minor planets on account of want of convergence of the series. For a long time astronomers had to be content with finding what are called the special perturbations of these bodies. And it was not until the brilliant researches of HANSEN on this subject that serious hopes were entertained of being able to find also the general perturbations of the minor planets. HANSEN'S mode of treatment differs entirely from those that had been previously employed. Instead of determining the perturbations of the rectangular or polar coördinates, or determining the variations of the elements of the orbit, he regards these elements as constant and finds what may be termed the perturbation of the time. The publication of his work, in which this new mode of treatment is given, entitled *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten*

Störungen der kleinen Planeten, undoubtedly marks a great advance in the determination of the general perturbations of the heavenly bodies.

The value of the work is greatly enhanced by an application of the method to a numerical example in which are given the perturbations of Egeria produced by the action of Jupiter, Mars, and Saturn. And yet, notwithstanding the many exceptional features of the work commending it to attention, astronomers seem to have been deterred by the refined analysis and laborious computations from anything like a general use of the method; and they still adhere to the method of special perturbations developed by LAGRANGE. HANSEN himself seems to have felt the force of the objections to his method, since in a posthumous memoir published in 1875, entitled *Ueber die Störungen der grossen Planeten, insbesondere des Jupiters*, his former positive views relative to the convergence of series, and the proper angles to be used in the arguments, are greatly modified.

HILL, in his work, *A New Theory of Jupiter and Saturn*, forming Vol. IV of the Astronomical Papers of the American Ephemeris, has employed HANSEN'S method in a modified form. In this work the author has given formulæ and developments of great utility when applied to calculations relating to the minor planets, and free use has been made of them in the present treatise. With respect to modifications in HANSEN'S original method made by that author himself, by HILL and others, it is to be noted that they have been made mainly, if not entirely, with reference to their employment in finding the general perturbations of the major planets.

The first use made of the method here given was for the purpose of comparing the values of the reciprocal of the distance and its odd powers as determined by the process of this paper, with the same quantities as derived according to HANSEN'S method. Upon comparison of the results it was found that the agreement was practically complete. To illustrate the application of his formulæ, HANSEN used Egeria whose eccentricity is comparatively small, being about $\frac{1}{12}$. The planet first chosen to test the method of this paper has an eccentricity of nearly $\frac{1}{7}$. And although the eccentricity in the latter planet was considerably larger, the convergence of the series in both methods was practically the same. It was then decided to test the adaptability of the method to the remaining steps of the problem, and the result of the work has been the preparation of the present paper.

HANSEN first expresses the odd powers of the reciprocal of the distance between the planets in series in which the angles employed are both eccentric anomalies. He then transforms the series into others in which one of the angles is the mean anomaly of the disturbing body. He makes still another transformation of his series so as to be able to integrate them.

In the method of this paper we at first employ the mean anomaly of the disturbed and the eccentric anomaly of the disturbing body, and as soon as we have the expressions for the odd powers of the reciprocal of the distance between the bodies, we make one transformation so as to have the mean anomalies of both planets in the arguments. These angles are retained unchanged throughout the subsequent work, enabling us to perform integration at any stage of the work.

In the expressions for the odd powers of the reciprocal of the distance we have, in the present method, the La Place coefficients entering as factors in the coefficients of the various arguments. These coefficients have been tabulated by RUNKLE in a work published by the SMITHSONIAN INSTITUTION entitled *New Tables for Determining the Values of the Coefficients in the Perturbative Function of Planetary Motion;* and hence the work relating to the determination of the expressions for the odd powers of the reciprocal of the distance is rendered comparatively short and simple.

In the expression for Δ^2, the square of the distance, the true anomaly is involved In the analysis we use the equivalent functions of the eccentric anomaly for those of the true anomaly, and when making the numerical computations we cause the eccentric anomaly of the disturbed body to disappear. This is accomplished by dividing the circumference into a certain number of equal parts relative to the mean anomaly and employing for the eccentric anomaly its numerical values corresponding to the various values of the mean anomaly.

Having the expressions for the odd powers of the reciprocal of the distance in series in which the angles are the mean anomaly of the disturbed body and the eccentric anomaly of the disturbing body, we derive, in Chapter II, expressions for the J or Besselian functions needed in transforming the series found into others in which both the angles will be mean anomalies.

In Chapter III expressions for the determination of the perturbing function and the perturbing forces are given. Instead of using the force involving the true anomaly we employ the one involving the mean anomaly. The disturbing forces employed are those in the direction of the disturbed radius-vector, in the direction perpendicular to this radius-vector, and in the direction perpendicular to the plane of the orbit.

Having the forces we then find the function W by integrating the expression

$$\frac{dW}{n\,.\,dt} = A\,.\,a\,\frac{d\Omega}{dg} + B\,.\,ar\,.\,\frac{d\Omega}{dr},$$

in which A, and B are factors easily determined.

From the value of W we derive that of $\overline{W}$ by simple mechanical processes, and then the perturbations of the mean anomaly and of the radius-vector are found from

$$n \,.\, \delta z = n \int \overline{W} \,.\, dt$$

$$\nu = - \tfrac{1}{2} n \int \frac{d \overline{W}}{d\gamma} \,.\, dt,$$

γ being a particular form for g.

The perturbation of the latitude is given by integrating the equation

$$\frac{d \,.\, \frac{u}{\cos i}}{n \,.\, dt} = C \,.\, a^2 \frac{d\Omega}{dZ},$$

C being a factor found in the same manner that A and B were.

It will be noticed that in finding the value of $n \,.\, \delta z$ two integrations are needed; in finding the perturbation of the latitude only one is required.

The arbitrary constants introduced by these integrations are so determined that the perturbations become zero for the epoch of the elements.

In all the applications of the method of this paper to different planets the circumference has been divided into sixteen parts, and the convergence of the different series is all that can be desired. In computing the perturbations of those of the minor planets whose eccentricities and inclinations are quite large, it may be necessary to divide the circumference into a larger number of parts. In exceptional cases, such as for Pallas, it may be necessary to divide the circumference into thirty-two parts.

In the different chapters of this paper the writer has given all that he conceives necessary for a full understanding of all the processes as they are in turn applied And he thinks there is nothing in the method here presented to deter any one with fair mathematical equipment from obtaining a clear idea of the means by which astronomers have been enabled to attain to their present knowledge of the motions of the heavenly bodies. The object always kept in mind has been to have at hand, in convenient form for reference and for application, the whole subject as it has been treated by HANSEN and others. Thus in connection with HANSEN's derivation of the function W, to obtain clearer conceptions of some matters presented, the method of BRÜNNOW for obtaining the same function has also been given. In some stages of the work where the experience of the writer has shown the need of particular care the work is

given with some detail. And while the writer is fully aware that here he may have exposed himself to criticism, it will suffice to state that he has not had in mind those competent of doing better, but rather the large class of persons that seems to have been deterred thus far, by imposing and formidable-looking formulæ, from becoming acquainted with the means and methods of theoretical astronomy. In the present state of the science there is greatly needed a large body of computers and investigators, so as to secure a fair degree of mastery over the constantly growing material.

The numerical example presented with the theory for the purpose of illustrating the new method will be found to cover a large part of the treatise. The example is designed to make evident the main steps and stages of the work, especially where these are left in any obscurity by the formulæ themselves. As a rule, the formulæ are given immediately in connection with their application and not merely by reference. It has been the wish to make this part of the treatise helpful to all who desire to exercise themselves in this field, and especially to those who desire to equip themselves for performing similar work.

The time required to determine the perturbations of a planet according to the method here given is believed to be very much less than that required by the unmodified method of HANSEN. Nearly all the time consumed in making the transformations by his mode of proceeding is here saved. The coefficients $b^{(i)}$ are much more quickly and readily found by making use of the tables prepared by RUNKLE, giving the values of these quantities. Doubtless experience will suggest still shorter processes than some of those here given and thus bring the subject within narrower limits in respect to the time required. If we compare the time demanded for the computation of the perturbations of the first order, with respect to the mass, produced by Jupiter, with the time needed to correct the elements after a dozen or more oppositions of the planet, computing three theoretical positions for each opposition, it is believed there will not be much difference, if any, in favor of the latter.

Again, when we wish to find only the perturbations of the first order, experience will show where many abridgments may safely be made. And whenever the positions of these bodies are made to depend upon those of comparison stars whose places are often not well determined, it will be found that the quality of the observed data does not justify refinements of calculation.

One of the things most needed in the theory of the motions of the minor planets is a general analytical expression for the perturbing function which may be applicable to all these small bodies. Thus if we had given the value of $a\Omega$ in terms of a periodic series, with literal coefficients and with the mean anomalies of the planets as the argu-

ments, we would at once have $a\frac{d\Omega}{dg}$ by differentiation. And since

$$r\frac{d\Omega}{dr} = a\frac{d\Omega}{da},$$

only two multiplications would be needed in finding the value of $\frac{dW}{n \,.\, dt}$, whose expression has been given above.

In the present paper we have dealt only with the perturbations of the first order with respect to the mass. The method has been employed in determining those of the second order also for two of the minor planets; but as those of Althæa, the planet employed in our example, have not yet been found, it was thought best not to give anything on the subject of the perturbations of the second order, until the perturbations of this order, in case of this body, are known.

The writer desires here to record his obligations to Prof. Edgar Frisby, of the U. S. Naval Observatory, Washington, D. C., and to Prof. George C. Comstock, Director of the Washburne Observatory, Madison, Wis., for kindly furnishing him with observations of planets that had not recently been observed; to Mr. Cleveland Keith, Assistant in the office of the American Ephemeris, for most valuable assistance in securing copies of observed places. And to Prof. Monroe B. Snyder, Director of the Central High School Observatory, Philadelphia, he is under special obligations for the interest manifested in the publication of this work, and for continued aid and most valuable suggestions in getting the work through the press.

CHAPTER I.

Development of the Reciprocal of the Distance Between the Planets and its Odd Powers in Periodic Series.

The action of one body on another under the influence of the law of gravitation is measured by the mass divided by the square of the distance. If then Δ be the distance between any two bodies, this distance varying from one instant to another, it will be necessary to find a convenient expression for $\left(\frac{1}{\Delta}\right)^2$ in terms of the time. If r and r' be the radii-vectores of the two bodies, the accented letter always referring to the disturbing body, we have

$$\Delta^2 = r^2 + r'^2 - 2rr' H.$$

If we introduce the semi-major axes a, a', which are constants, and their relation $\alpha = \frac{a'}{a}$, we obtain

$$\left(\frac{\Delta}{a}\right)^2 = \left(\frac{r}{a}\right)^2 + \left(\frac{r'}{a'}\right)^2 \alpha^2 - 2\left(\frac{r}{a}\right)\left(\frac{r'}{a'}\right) \alpha H, \tag{1}$$

H being the cosine of the angle formed by the radii-vectores.

Let the origin of angles be taken at the ascending node of the plane of the disturbed, on the plane of the disturbing, body. Let Π, Π', be the longitudes of the perihelia measured from this point; also let f, f', be the true anomalies. The angle formed by the radii-vectores is $(f' + \Pi') - (f + \Pi)$; and the angles $f + \Pi, f + \Pi'$, being in different planes, we have

$$H = \cos(f + \Pi)\cos(f' + \Pi') + \cos I \sin(f + \Pi)\sin(f' + \Pi'), \tag{2}$$

I being the mutual inclination of the two planes.

To find the values of Π, Π', I, let Φ be the angular distance from the ascending node of the plane of the disturbed body on the fundamental plane to its ascending

node on the plane of the disturbing body. Let ψ be the angular distance from ascending node of the plane of the disturbing body on the fundamental plane to the same point.

If π, π', are the longitudes of the perihelia,

Ω, Ω', the longitudes of the ascending nodes on the fundamental plane adopted, which is generally that of the ecliptic, we have

$$\Pi = \pi - \Omega - \Phi, \qquad \Pi' = \pi' - \Omega' - \psi. \tag{3}$$

The angles Φ, ψ, $\Omega - \Omega'$, are the sides of a spherical triangle, lying opposite the angles i', $180 - i$, I,

i, i', being the inclination of disturbed and disturbing body on the fundamental plane.

The angles I, Φ, ψ, are found from the equations

$$\left.\begin{aligned}
\sin \tfrac{1}{2} I \sin \tfrac{1}{2} (\psi + \Phi) &= \sin \tfrac{1}{2} (\Omega - \Omega') \sin \tfrac{1}{2} (i' + i) \\
\sin \tfrac{1}{2} I \cos \tfrac{1}{2} (\psi + \Phi) &= \cos \tfrac{1}{2} (\Omega - \Omega') \sin \tfrac{1}{2} (i - i') \\
\cos \tfrac{1}{2} I \sin \tfrac{1}{2} (\psi - \Phi) &= \sin \tfrac{1}{2} (\Omega - \Omega') \cos \tfrac{1}{2} (i' + i) \\
\cos \tfrac{1}{2} I \cos \tfrac{1}{2} (\psi - \Phi) &= \cos \tfrac{1}{2} (\Omega - \Omega') \cos \tfrac{1}{2} (i - i')
\end{aligned}\right\} \tag{4}$$

In using these equations when Ω is less than Ω' we must take $\frac{1}{2}(360° + \Omega - \Omega')$ instead of $\frac{1}{2}(\Omega - \Omega')$.

We have a check on the values of I, Φ, ψ, by using the equations given in HANSEN's posthumous memoir, p. 276.

Thus we have

$$\left.\begin{aligned}
\cos p \,.\, \sin q &= \sin i' \,.\, \cos (\Omega - \Omega') \\
\cos p \,.\, \cos q &= \cos i' \\
\cos p \,.\, \sin r &= \cos i' \,.\, \sin (\Omega - \Omega') \\
\cos p \,.\, \cos r &= \cos (\Omega - \Omega') \\
\sin p &= \sin i' \sin (\Omega - \Omega') \\
\sin I \sin \Phi &= \sin p \\
\sin I \cos \Phi &= \cos p \,.\, \sin (i - q) \\
\sin I \sin (\psi - r) &= \sin p \,.\cos (i - q) \\
\sin I \cos (\psi - r) &= \sin (i - q) \\
\cos I &= \cos p \,.\, \cos (i - q)
\end{aligned}\right\} \tag{5}$$

To develop the expression for $\left(\frac{\Delta}{a}\right)$, we put

$$\left.\begin{aligned} \cos I . \sin \Pi' = k \sin K, \qquad \sin \Pi' = k_1 \sin K_1, \\ \cos \Pi' = k \cos K, \cos I \cos \Pi' = k_1 \cos K_1, \end{aligned}\right\} \tag{6}$$

and hence

$$\begin{aligned} H = \cos f . \cos f' . k \cos (\Pi - K) + \cos f . \sin f' . k_1 \sin (\Pi - K_1) \\ - \sin f . \cos f' . k \sin (\Pi - K) + \sin f . \sin f' . k_1 \cos (\Pi - K_1). \end{aligned}$$

Introducing the eccentric anomaly ε, we have

$$\cos f = \frac{a}{r} (\cos \varepsilon - e), \qquad \sin f = \frac{a}{r} . \cos \phi . \sin \varepsilon,$$

e being the eccentricity, and ϕ the angle of eccentricity; and find

$$\begin{aligned} \frac{r}{a} . \frac{r'}{a'} . H = & \cos \varepsilon . \cos \varepsilon' . k \cos (\Pi - K) - \cos \varepsilon' . ek \cos (\Pi - K) \\ & - \cos \varepsilon . e'k \cos (\Pi - K) + ee'k \cos (\Pi - K) \\ & + \cos \varepsilon . \sin \varepsilon' . \cos \phi' . k_1 \sin (\Pi - K_1) - \sin \varepsilon' . e . \cos \phi' . k_1 \sin (\Pi - K_1) \\ & - \sin \varepsilon . \cos \varepsilon' . \cos \phi . k \sin (\Pi - K) + \sin \varepsilon . e' . \cos \phi . k \sin (\Pi - K) \\ & + \sin \varepsilon . \sin \varepsilon' . \cos \phi . \cos \phi' . k_1 \cos (\Pi - K_1). \end{aligned}$$

Substituting the value of $\frac{r}{a} . \frac{r'}{a'} . H$ in the expression for $\left(\frac{\Delta}{a}\right)^2$ we have

$$\begin{aligned} \left(\frac{\Delta}{a}\right)^2 = & 1 + \alpha^2 - 2e . \cos \varepsilon + e^2 \cos^2 \varepsilon - 2 \alpha ee'k \cos (\Pi - K) \\ & + 2\alpha e'k \cos (\Pi - K) \cos \varepsilon - 2\alpha e' \cos \phi . k \sin (\Pi - K) \sin \varepsilon \\ & - [2\alpha^2 e' - 2\alpha ek \cos (\Pi - K) + 2\alpha k \cos (\Pi - K) \cos \varepsilon \\ & - 2\alpha \cos \phi . k \sin (\Pi - K) \sin \varepsilon] . \cos \varepsilon' \\ & - [- 2\alpha e \cos \phi' . k_1 \sin (\Pi - K_1) + 2\alpha \cos \phi \cos \phi' . k_1 \cos (\Pi - K_1) \sin \varepsilon \\ & + 2\alpha \cos \phi' . k_1 \sin (\Pi - K_1) \cos \varepsilon] . \sin \varepsilon' \\ & + \alpha^2 e'^2 . \cos^2 \varepsilon'. \end{aligned}$$

Putting γ_1, β_0, γ_2, for the coefficients of $\cos \varepsilon'$, $\sin \varepsilon'$, $\cos^2 \varepsilon'$, respectively, and γ_0 for the term not affected by $\cos \varepsilon'$ or $\sin \varepsilon'$, we have the abbreviated form

$$\left(\frac{\Delta}{a}\right)^2 = \gamma_0 - \gamma_1 . \cos \varepsilon' - \beta_0 . \sin \varepsilon' + \gamma_2 . \cos^2 \varepsilon'. \tag{7}$$

In this expression for $\left(\frac{\Delta}{a}\right)^2$, γ_0, γ_1, and β_0 are functions of the eccentric anomaly of the disturbed body; γ_2 is a constant and of the order of the square of the eccentricity of the disturbing body.

In the method here followed the circumference in case of the disturbed body will be divided into a certain number of equal parts with respect to the mean anomaly, g. The various values of g will then be $0°$, $\frac{360°}{n}$, $2.\frac{360°}{n}$, $3.\frac{360°}{n}$, $\ldots\ldots$ $n-1.\frac{360°}{n}$.

For each numerical value of g, the corresponding value of ε is found from

$$g = \varepsilon - e \sin \varepsilon.$$

Before substituting the numerical values of $\cos \varepsilon$, $\sin \varepsilon$, for the n divisions of the circumference, the expressions for γ_0, γ_1, β_0, will be put in a form most convenient for computation.

Let

$$\left.\begin{aligned} p.\sin P &= 2\alpha^2 \frac{e'}{e} - 2\alpha k \cos (\Pi - K) \\ p.\cos P &= 2\alpha \cos \phi' k_1 \sin (\Pi - K_1), \end{aligned}\right\} \qquad (8)$$

and

$$\left.\begin{aligned} \beta_0 &= f.\sin F \\ \gamma_1 &= f.\cos F; \end{aligned}\right\} \qquad (9)$$

we find

$$\beta_0 = f \sin F = 2\alpha \,.\cos \phi \,.\cos \phi' .\, k_1 \cos (\Pi - K_1).\sin \varepsilon + p \cos P.\cos \varepsilon - ep\,.\cos P$$
$$\gamma_1 = f \cos F = \left(2\alpha^2 \frac{e'}{e} - p \sin P\right).\cos \varepsilon - 2\alpha\,.\cos \phi\,.\,k \sin (\Pi - K)\,.\sin \varepsilon + ep\,.\sin P.$$

And from these equations we find, since

$$f.\sin (F - P) = f.\sin F \cos P - f \cos F.\sin P$$
$$f.\cos (F - P) = f \cos F.\cos P + f \sin F.\sin P,$$

$$f.\sin (F - P) = [2\alpha\,.\cos \phi\,.\cos \phi'.\,k_1 \cos (\Pi - K_1).\cos P$$
$$+ 2\alpha.\cos \phi.k \sin(\Pi - K).\sin P].\sin \varepsilon + \left[p - 2\alpha^2 \frac{e'}{e} \sin P\right].\cos \varepsilon - ep$$

$$f.\cos (F - P) = [2\alpha\,.\cos \phi\,.\cos \phi'.\,k_1 \cos (\Pi - K_1)\,.\sin P$$
$$- 2\alpha\,.\cos \phi\,.\,k \sin (\Pi - K)\,.\cos P]\,.\sin \varepsilon + 2\alpha^2.\frac{e'}{e}.\cos P\,.\cos \varepsilon$$

If we now put

$$\left.\begin{aligned}
v \sin V &= 2\alpha \,.\, \cos\phi \,.\, k \sin(\Pi - K) \\
v \cos V &= 2\alpha \,.\, \cos\phi \,.\, \cos\phi' \,.\, k_1 \cos(\Pi - K_1) \\
w \sin W &= p - 2\alpha^2 \,.\, \frac{e'}{e} \,.\, \sin P \\
w \cos W &= v \,.\, \cos(V - P) \\
w_1 \sin W_1 &= v \,.\, \sin(V - P) \\
w_1 \cos W_1 &= 2\alpha^2 \,.\, \frac{e'}{e} \,.\, \cos P,
\end{aligned}\right\} \tag{10}$$

we get

$$\begin{aligned}
f \,.\, \sin(F - P) &= w \,.\, \sin(\varepsilon + W) - ep \\
f \,.\, \cos(F - P) &= w_1 \,.\, \cos(\varepsilon + W_1).
\end{aligned} \tag{11}$$

Further, if we put

$$R = 1 + \alpha^2 - 2\alpha^2 \,.\, e'^2, \tag{12}$$

we have

$$\gamma_0 = R - 2e \,.\, \cos\varepsilon + e^2 \,.\, \cos^2\varepsilon + e'\gamma_1$$

or,

$$\gamma_0 = R - 2e \,.\, \cos\varepsilon + e^2 \,.\, \cos^2\varepsilon + e' \,.\, f \cos F. \tag{13}$$

We find the value of γ_2 from

$$\gamma_2 = \alpha^2 \,.\, e'^2.$$

The constants, k, K, k_1, K_1, p, P, w, W, w_1, W_1, R, are found, once for all, from the equations given above. For every value of ε we have the corresponding value of f and F from equations (11); hence, also the values of $f \sin F$, $f \cos F$, which are the values of β_0 and γ_1. Equation (13) furnishes the value of γ_0 by substituting in it the various numerical values of ε, as was done for β_0 and γ_1. The value of the coefficient γ_2 being constant, we thus have given the values of $\left(\frac{\Delta}{a}\right)^2$ for as many points along the circumference as there are divisions.

We can put

$$\left(\frac{\Delta}{a}\right)^2 = \gamma_0 - \gamma_1 \cos \varepsilon' - \beta_0 . \sin \varepsilon' + \gamma_2 . \cos .^2 \varepsilon'$$

in the form

$$\left(\frac{\Delta}{a}\right)^2 = [C - q . \cos (\varepsilon' - Q)]\ [1 - q_1 . \cos (\varepsilon' - Q_1)], \tag{14}$$

in which the factor $1 - q_1 . \cos (\varepsilon' - Q_1)$ differs little from unity. For this purpose, if we perform the operations indicated in the second expression, and then compare the coefficients of like terms, we find

$$\begin{aligned}
\gamma_0 &= C + q . q_1 \sin Q . \sin Q_1 \\
\gamma_1 &= q . \cos Q + q_1 . C \cos Q_1 \\
\gamma_2 &= q . q_1 . \cos (Q + Q_1) \\
\beta_0 &= q . \sin Q + q_1 . C \sin Q_1 \\
0 &= \sin (Q + Q_1).
\end{aligned}$$

The last of these equations is satisfied by putting

$$Q_1 = - Q.$$

The remaining equations then take the form

$$\left.\begin{aligned}
\gamma_0 &= C - q . q_1 . \sin^2 Q \\
\gamma_1 &= (q + q_1 . C) . \cos Q \\
\gamma_2 &= q . q_1 \\
\beta_0 &= (q - q_1 . C) . \sin Q
\end{aligned}\right\} \tag{15}$$

The expressions

$$\left.\begin{aligned}
q . \sin Q &= \beta_0 + \xi \\
q . \cos Q &= \gamma_1 - \eta \\
q_1 . C . \sin Q &= \xi \\
q_1 . C . \cos Q &= \eta
\end{aligned}\right\} \tag{16}$$

satisfy the relations expressed by the second and fourth of equations (15), where $C = \gamma_0 + \zeta$.

We have now to find expressions for the small quantities ξ, η, ζ found in these equations.

Equations (16) give

$$q \cdot q_1 \cdot C \sin^2 Q = (\beta_0 + \xi) \cdot \xi.$$

The equation

$$\gamma_0 = C - q \cdot q_1 \sin^2 Q$$

then becomes

$$(\gamma_0 + \zeta)\,\zeta = (\beta_0 + \xi)\,\xi \tag{a}$$

From (16) we have, also,

$$q \cdot q_1 \cdot C = (\beta_0 + \xi)\,\xi + (\gamma_1 - \eta)\,\eta,$$

from which, since $\gamma_2 = q \cdot q_1$, and $C = \gamma_0 + \zeta$, we obtain

$$(\gamma_0 + \zeta) \cdot \gamma_2 = (\beta_0 + \xi)\,\xi + (\gamma_1 - \eta)\,\eta. \tag{b}$$

Equations (16) give again

$$(\gamma_1 - \eta)\,\xi = (\beta_0 + \xi)\,\eta. \tag{c}$$

When ζ is known, ξ is found from (a); and the difference between (a) and (b)

$$(\gamma_0 + \zeta)\,(\gamma_2 - \zeta) = (\gamma_1 - \eta) \cdot \eta \tag{d}$$

gives η when ζ is known.

The equations (a) and (c) give

$$\beta_0^2 + 4\,(\gamma_0 + \zeta)\,\zeta = (\beta_0 + 2\,\xi)^2$$
$$\beta_0 + 2\,\xi = \gamma_1 \cdot \frac{\xi}{\eta};$$

and hence

$$\beta_0^2 + 4\,(\gamma_0 + \zeta)\,\zeta = \gamma_1^2 \cdot \frac{\xi^2}{\eta^2}$$

Deduce the values of $\beta_0 + \xi$, $\gamma_1 - \eta$ from (a) and (d), substitute them in (c), we find

$$\frac{\eta^2}{\xi^2} = \frac{\gamma_2 - \zeta}{\zeta}.$$

The last equation then takes the form

$$0 = \gamma_1^2 . \zeta - \beta_0^2 (\gamma_2 - \zeta) - 4 (\gamma_0 + \zeta)(\gamma_2 - \zeta) . \zeta. \tag{e}$$

This equation furnishes the value of ζ; and with ζ known, we find ξ, η, from equations already given. The three equations giving the values of the quantities sought are

$$\left.\begin{aligned} \zeta^3 + (\gamma_0 - \gamma_2)\zeta^2 + \tfrac{1}{4}(\gamma_1^2 + \beta_0^2 - 4\gamma_0 . \gamma_2)\zeta - \tfrac{1}{4} . \beta_0^2 . \gamma_2 &= 0 \\ \xi^2 + \beta_0 . \xi - (\gamma_0 + \zeta)\zeta &= 0 \\ \eta^2 - \gamma_1 . \eta + (\gamma_0 + \zeta)(\gamma_2 - \zeta) &= 0 \end{aligned}\right\} \tag{f}$$

Finding the values of ζ, ξ, η, from these equations, and arranging with respect to γ_2, preserving only the first power, we have

$$\left.\begin{aligned} \zeta &= \frac{\beta_0^2}{\gamma_1^2 + \beta_0^2} . \gamma_2 \\ \xi &= \frac{\gamma_0 . \beta_0}{\gamma_1^2 + \beta_0^2} . \gamma_2 \\ \eta &= \frac{\gamma_0 . \gamma_1}{\gamma_1^2 + \beta_0^2} . \gamma_2 \end{aligned}\right\} \tag{g}$$

Substituting these values in equations (16), they become

$$\left.\begin{aligned} q . \sin Q &= \beta_0 + \frac{\gamma_0 . \beta_0}{\gamma_1^2 + \beta_0^2} . \gamma_2 \\ q . \cos Q &= \gamma_1 - \frac{\gamma_0 . \gamma_1}{\gamma_1^2 + \beta_0^2} . \gamma_2 \\ q_1\, C \sin Q &= \frac{\gamma_0 . \beta_0}{\gamma_1^2 + \beta_0^2} . \gamma_2 \\ q_1\, C \cos Q &= \frac{\gamma_0 . \gamma_1}{\gamma_1^2 + \beta_0^2} . \gamma_2 \end{aligned}\right\} \tag{17}$$

noting that $C = \gamma_0 + \zeta$.

If more accurate values of ζ, ξ, η, are needed than those given by equations (g), we proceed as follows:

Substitute the value of ζ given by (g) in the second term of the first of equations (f), we find, up to terms including γ_2^2,

$$\zeta = \frac{\beta_0^2}{\gamma_1^2 + \beta_0^2} . \gamma_2 + 4 . \frac{\gamma_0 . \beta_0^2}{(\gamma_1^2 + \beta_0^2)^2} . \gamma_2^2 - 4 . \frac{\gamma_0 . \beta_0^4}{(\gamma_1^2 + \beta_0^2)^3} . \gamma_2^2. \tag{18}$$

The last two of (f) give also

$$\xi = \frac{C \cdot \zeta}{\beta_0} - \frac{C^2 \cdot \zeta^2}{\beta_0^{\,3}}$$
$$\eta = \frac{C(\gamma_2 - \zeta)}{\gamma_1} + \frac{C^2(\gamma_2 - \zeta)^2}{\gamma_1^{\,3}}$$

Introducing the values of f, F, given by (11), putting

$$\chi = \gamma_2 + 4 \cdot \gamma_2^{\,2} \cdot \frac{\gamma_0}{f^2} \cdot \cos^2 F$$
$$\chi' = \gamma_2 - 4 \cdot \gamma_2^{\,2} \cdot \frac{\gamma_0}{f^2} \cdot \sin^2 F \tag{19}$$

we have

$$\zeta = \chi \cdot \sin^2 F,$$

so that

$$C = \gamma_0 + \chi \cdot \sin^2 F. \tag{20}$$

Moreover, since

$$\gamma_2 - \zeta = \chi' \cdot \cos^2 F,$$

we find from the expressions for ξ, η, given above,

$$\beta_0 + \xi = f \cdot \xi' \cdot \sin F,$$
$$\gamma_1 - \eta = f \cdot \eta' \cdot \cos F,$$

if

$$\xi' = 1 + \frac{C\chi}{f^2} - \left(\frac{C\chi}{f^2}\right)^2$$
$$\eta' = 1 - \frac{C\chi'}{f^2} - \left(\frac{C\chi'}{f^2}\right)^2 \tag{21}$$

Substituting these in the expressions for $q \sin Q$, $q \cos Q$, they become

$$q \sin Q = f \cdot \xi' \cdot \sin F$$
$$q \cos Q = f \cdot \eta' \cdot \cos F. \tag{22}$$

The value of q_1 is found from

$$q_1 = \frac{r_2}{q} \tag{23}$$

The quantities q, q_1, Q can be expressed in another manner. The equations (22) give

$$tg\ Q = \frac{\xi'}{\eta'} . tg\ F$$
$$q^2 = f^2 . \xi'^2 . \sin{}^2 F + f^2 . \eta'^2 . \cos{}^2 F;$$

from which we derive

$$Q = F + \frac{\xi' - \eta'}{\xi' + \eta'} . \sin 2F + \tfrac{1}{2} \left(\frac{\xi' - \eta'}{\xi' + \eta'}\right)^2 . \sin 4F + \text{etc.}$$
$$\log. q = \log. f + \tfrac{1}{2} \log. (\xi'^2 . \sin{}^2 F + \eta'^2 \cos{}^2 F).$$

Since χ^2 and χ'^2 agree up to terms of the third order, the equations for ξ' and η' give

$$\frac{\xi' - \eta'}{\xi' + \eta'} = \frac{C(\chi + \chi')}{2f^2};$$

or,

$$\frac{\xi' - \eta'}{\xi' + \eta'} = \frac{r_0 r_2}{f^2} + \frac{r_2^2}{2f^2} + \left(2\frac{r_0^2 r_2^2}{f^4} - \frac{r_2^2}{2f^2}\right) \cos 2F$$

Further

$$\xi'^2 \sin{}^2 F + \eta'^2 \cos{}^2 F = 1 + 2\frac{C}{f^2}(\chi . \sin{}^2 F - \chi' \cos{}^2 F) - \left(\frac{C\chi}{f^2}\right)^2$$

and

$$\tfrac{1}{2} \log. (\xi'^2 \sin{}^2 F + \eta'^2 \cos{}^2 F) = \frac{C}{f^2}(\chi \sin{}^2 F - \chi' \cos{}^2 F)$$
$$- \frac{C^2}{f^4}(\chi \sin{}^2 F - \chi' \cos{}^2 F)^2 - \tfrac{1}{2}\left(\frac{C\chi}{f^2}\right)^2$$

Substituting the values of χ, χ', C, given before, we find

$$\frac{C}{f^2}(\chi \sin{}^2 F - \chi' \cos{}^2 F) = \frac{r_0^2 r_2^2}{f^4} + \frac{r_2^2}{4f^2} - \left(\frac{r_0 r_2}{f^2} + \frac{r_2^2}{2f^2}\right) \cos 2F$$
$$- \left(\frac{r_0^2 r_2^2}{f^4} - \frac{r_2^2}{4f^2}\right) \cos 4F$$

$$\left(\frac{C\chi}{f^2}\right)^2 = \frac{r_0^2 r_2^2}{f^4}$$

The equation $\gamma_2 = q \cdot q_1$ gives

$$\log. \gamma_2 = \log. q + \log. q_1$$

Putting

$$\log. q = \log. f + y,$$

we have for q_1

$$\log. q_1 = \log. \frac{\gamma_2}{f} - y.$$

Writing s for the number of seconds in the radius, and λ_0 for the modulus of the common system of logarithms, we find

$$\left.\begin{aligned} Q &= F + x \\ \log. q &= \log. f + y \\ \log. q_1 &= \log. \frac{\gamma_2}{f} - y \end{aligned}\right\} \qquad (24)$$

in which

$$\left.\begin{aligned} x &= \qquad\qquad s\left(\frac{\gamma_0 \gamma_2}{f^2} + \frac{\gamma_2^2}{2f^2}\right) \sin 2F + s\left(\frac{3\gamma_0^2 \gamma_2^2}{2f^4} - \frac{\gamma_2^2}{4f^2}\right) \sin 4F \\ y &= \lambda_0 \frac{\gamma_2^2}{4f^2} - \lambda_0 \left(\frac{\gamma_0 \gamma_2}{f^2} + \frac{\gamma_2^2}{2f^2}\right) \cos 2F - \lambda_0 \left(\frac{3\gamma_0^2 \gamma_2^2}{2f^4} - \frac{\gamma_2^2}{4f^2}\right) \cos 4F \end{aligned}\right\} \qquad (25)$$

And for C we have from the first of (15)

$$C = \gamma_0 + \gamma_2 \cdot \sin^2 Q. \qquad (26)$$

By means of the last three equations we are enabled to find the values of Q, q, q_1, C, with the greatest accuracy. The equations (17), where not sufficiently approximate, will, nevertheless, furnish a good check on the values of these quantities. All the quantities in the expression for $\left(\frac{\Delta}{a}\right)^2$ are thus known; and substituting their values corresponding to the various values of g, we have the values of $\left(\frac{\Delta}{a}\right)^2$ for the different points of the circumference.

Using the values of C, q, q_1, Q, just found, HILL, in his *New Theory of Jupiter and Saturn*, has given another expression for $\left(\frac{\Delta}{a}\right)$ which we shall employ.

To transform

$$\left(\frac{\Delta}{a}\right)^2 = \left(C - q \cdot \cos\left(\varepsilon' - Q\right)\right)\left(1 - q_1 \cdot \cos\left(\varepsilon' + Q\right)\right)$$

into the required form we put

$$\left.\begin{aligned} \frac{q}{C} &= \sin\chi, \quad \frac{\gamma_2}{q} = \sin\chi_1 \\ a &= tg\,\tfrac{1}{2}\chi, \quad b = tg\,\tfrac{1}{2}\chi_1 \\ N &= \frac{\sec\frac{1}{2}\chi \cdot \sec\frac{1}{2}\chi_1}{\sqrt{C}} \end{aligned}\right\} \qquad (27)$$

Then

$$\begin{aligned} \left(\frac{\Delta}{a}\right)^2 &= C\left[1 - \sin\chi \cdot \cos\left(\varepsilon' - Q\right)\right]\left[1 - \sin\chi_1 \cdot \cos\left(\varepsilon' + Q\right)\right] \\ &= \frac{C\left[\sec^2\frac{1}{2}\chi\left(1 - \sin\chi \cdot \cos\left(\varepsilon' - Q\right)\right)\right]}{\sec^2\frac{1}{2}\chi} \, \frac{\left[\sec^2\frac{1}{2}\chi_1\left(1 - \sin\chi_1 \cdot \cos\left(\varepsilon' + Q\right)\right)\right]}{\sec^2\frac{1}{2}\chi_1} \\ &= \frac{C\left[1 + tg^2\frac{1}{2}\chi - 2tg\,\frac{1}{2}\chi\cos\left(\varepsilon' - Q\right)\right]}{\sec^2\frac{1}{2}\chi} \, \frac{\left[1 + tg^2\frac{1}{2}\chi_1 - 2tg\,\frac{1}{2}\chi_1\cos\left(\varepsilon' + Q\right)\right]}{\sec^2\frac{1}{2}\chi_1} \end{aligned}$$

Substituting the values of a, b, N, we get

$$\left(\frac{a}{\Delta}\right)^n = N^n\left[1 + a^2 - 2a\cos\left(\varepsilon' - Q\right)\right]^{-\frac{n}{2}}\left[1 + b^2 - 2b\cos\left(\varepsilon' + Q\right)\right]^{-\frac{n}{2}} \qquad (28)$$

We compute the values of a, b, N, corresponding to the different values of g, and check by finding the sums of the odd and the even orders, which should be nearly the same. If we put

$$\left[1 + a^2 - 2a\cos\left(\varepsilon' - Q\right)\right]^{-s} = \left[\tfrac{1}{2}\,b^{(0)} + b^{(1)} \cdot \cos\theta + b^{(2)} \cdot \cos 2\vartheta + b^{(3)} \cdot \cos 3\vartheta + \text{etc.}\right]$$

$$\left[1 + b^2 - 2b\cos\left(\varepsilon' + Q\right)\right]^{-s} = \left[\tfrac{1}{2}\,B^{(0)} + B^{(1)} \cdot \cos\left(\varepsilon' + Q\right) + B^{(2)} \cdot \cos 2\left(\varepsilon' + Q\right) + \text{etc.}\right]$$

where $s = \frac{n}{2}$, $\theta = \varepsilon' - Q$, we are enabled to make use of coefficients already known.

For $2 \,.\, \cos\theta$, write $x + \frac{1}{x}$, and then we have

$$\left[1 + a^2 - 2a\cos\theta\right]^{-s} = \left[1 + a^2 - a\left(x + \frac{1}{x}\right)\right]^{-s}$$
$$= \left[1 - ax\right]^{-s}\left[1 - \frac{a}{x}\right]^{-s}$$

Expanding we have

$$\left[1 - ax\right]^{-s} = 1 + \frac{s}{1} \,.\, ax + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, a^2x^2 + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, a^3x^3 + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, a^4x^4$$
$$+ \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, \frac{s+4}{5} \,.\, a^5x^5 + \text{etc.}$$

$$\left[1 - \frac{a}{x}\right]^{-s} = 1 + \frac{s}{1} \,.\, \frac{a}{x} + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{a^2}{x^2} + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{a^3}{x^3} + \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, \frac{a^4}{x^4}$$
$$+ \frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, \frac{s+4}{5} \,.\, \frac{a^5}{x^5} + \text{etc.}$$

And hence, for their product, we have

$$1 + \left(\frac{s}{1}\right)^2 \,.\, a^2 + \left(\frac{s}{1} \,.\, \frac{s+1}{2}\right)^2 \,.\, a^4 + \left(\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3}\right)^2 \,.\, a^6 + \left(\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4}\right)^2 \,.\, a^8 + \text{etc.}$$
$$+ \left[\frac{s}{1} \,.\, a + \left(\frac{s}{1}\right)^2 \,.\, \frac{s+1}{2} \,.\, a^3 + \left(\frac{s}{1} \,.\, \frac{s+1}{2}\right)^2 \,.\, \frac{s+2}{3} \,.\, a^5\right.$$
$$\left. + \left(\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3}\right)^2 \,.\, \frac{s+3}{4} \,.\, a^7 + \text{etc.}\right]\left(x + \frac{1}{x}\right)$$
$$+ \left[\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, a^2 + \left(\frac{s}{1}\right)^2 \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, a^4 + \left(\frac{s}{1} \,.\, \frac{s+1}{2}\right)^2 \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, a^6\right.$$
$$\left. + \left(\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3}\right)^2 \,.\, \frac{s+3}{4} \,.\, \frac{s+4}{5} \,.\, a^8 + \text{etc.}\right]\left(x^2 + \frac{1}{x^2}\right)$$
$$+ \left[\frac{s}{1} \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, a^3 + \left(\frac{s}{1}\right)^2 \,.\, \frac{s+1}{2} \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, a^5\right.$$
$$\left. + \left(\frac{s}{1} \,.\, \frac{s+1}{2}\right)^2 \,.\, \frac{s+2}{3} \,.\, \frac{s+3}{4} \,.\, \frac{s+4}{5} \,.\, a^7 + \text{etc.}\right]\left(x^3 + \frac{1}{x^3}\right)$$
$$+ \text{etc.}$$

But $x + \frac{1}{x} = 2\cos\theta$, $\quad x^2 + \frac{1}{x^2} = 2 \,.\, \cos 2\theta$, $\quad x^3 + \frac{1}{x^3} = 2 \,.\, \cos 3\theta$, $\quad$ etc.,

and hence

$$\left.\begin{aligned}
\tfrac{1}{2}b^{(0)} &= 1+\left(\tfrac{s}{1}\right)^2.a^2+\left(\tfrac{s}{1}.\tfrac{s+1}{2}\right)^2.a^4+\left(\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+2}{3}\right)^2.a^6+\text{etc.}\\
b^{(1)} &= 2sa\left[1+\tfrac{s}{1}.\tfrac{s+1}{2}.a^2+\tfrac{s}{1}.\left(\tfrac{s+1}{2}\right)^2.\tfrac{s+2}{3}.a^4+\tfrac{s}{1}.\left(\tfrac{s+1}{2}.\tfrac{s+2}{3}\right)^2.\tfrac{s+3}{4}.a^6+\text{etc.}\right]\\
b^{(2)} &= 2.\tfrac{s}{1}.\tfrac{s+1}{2}.a^2\Big[1+\tfrac{s}{1}.\tfrac{s+2}{3}.a^2+\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+2}{3}.\tfrac{s+3}{4}.a^4\\
&\qquad+\tfrac{s}{1}.\tfrac{s+1}{2}\left(\tfrac{s+2}{3}\right)^2.\tfrac{s+3}{4}.\tfrac{s+4}{5}.a^6+\text{etc.}\Big]\\
b^{(3)} &= 2.\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+2}{3}.a^3\Big[1+\tfrac{s}{1}.\tfrac{s+3}{4}.a^2+\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+3}{4}.\tfrac{s+4}{5}.a^4\\
&\qquad+\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+2}{3}.\tfrac{s+3}{4}.\tfrac{s+4}{5}.\tfrac{s+5}{6}.a^6+\text{etc.,}\Big]\\
&\text{and generally}\\
b^{(i)} &= 2.\tfrac{s}{1}.\tfrac{s+1}{2}\ldots\tfrac{(s+i-1)}{i}.a^i\left[1+\tfrac{s}{1}.\tfrac{s+i}{i+1}.a^2+\tfrac{s}{1}.\tfrac{s+1}{2}.\tfrac{s+i}{i+1}.\tfrac{s+i+1}{i+2}.a^4+\text{etc.}\right]
\end{aligned}\right\}\quad(29)$$

Since $s=\frac{n}{2}$, we find from these expressions the values of the $b^{(i)}$ coefficients for different values of n.

Runkle has tabulated the values of $b^{(i)}$ in a paper published by the Smithsonian Institution. Thus the value of

$$[1+a^2-2a\cos(\varepsilon'-Q)]^{-\frac{n}{2}}$$

is obtained with great facility.

The value of $[1+b^2-2b\cos(\varepsilon+Q)]^{-\frac{n}{2}}$ is found in the same way.

We now let

$$\left.\begin{aligned}
c^{(i)} &= \tfrac{1}{2}.N.B^{(i)}.\cos 2\,iQ\\
s^{(i)} &= \tfrac{1}{2}.N.B^{(i)}.\sin 2\,.\,iQ
\end{aligned}\right\}\quad(30)$$

And hence have

$$\begin{aligned}
c^{(0)} &= \tfrac{1}{2}.N.B^{(0)}\\
c^{(1)} &= \tfrac{1}{2}.N.B^{(1)}.\cos 2Q\\
s^{(1)} &= \tfrac{1}{2}.N.B^{(1)}.\sin 2Q\\
c^{(2)} &= \tfrac{1}{2}.N.B^{(2)}.\cos 4Q\\
s^{(2)} &= \tfrac{1}{2}.N.B^{(2)}.\sin 4Q\\
\text{etc.} &= \text{etc.}
\end{aligned}$$

Multiplying the series $[\frac{1}{2}b^{(0)} + b^{(1)}.\cos\theta + b^{(2)}.\cos 2\theta + b^{(3)}.\cos 3\theta + \text{etc.}]$
by $[\frac{1}{2}B^{(0)} + B^{(1)}\cos(\varepsilon' + Q) + B^{(2)}.\cos 2(\varepsilon' + Q) + \text{etc.}]$,

noting that $\theta = Q - \varepsilon'$, and arranging the terms with respect to $\cos i\theta$, $\sin i\theta$, we find

$$\left.\begin{aligned}
\left(\frac{a}{\varDelta}\right) &= \tfrac{1}{2}b^{(0)}.c^{(0)} + b^{(1)}.c^{(1)} + b^{(2)}.c^{(2)} \\
&+ [b^{(1)}.c^{(0)} + (b^{(0)} + b^{(2)})\,c^{(1)} + (b^{(1)} + b^{(3)})\,c^{(2)}]\cos\theta \\
&+ [\qquad\quad + (b^{(0)} - b^{(2)})\,s^{(1)} + (b^{(1)} - b^{(3)})\,s^{(2)}]\sin\theta \\
&+ [b^{(2)}.c^{(0)} + (b^{(1)} + b^{(3)})\,c^{(1)} + (b^{(0)} + b^{(4)})\,c^{(2)}]\cos 2\theta \\
&+ [\qquad\quad + (b^{(1)} - b^{(3)})\,s^{(1)} + (b^{(0)} - b^{(4)})\,s^{(2)}]\sin 2\theta \\
&+ [b^{(3)}.c^{(0)} + (b^{(2)} + b^{(4)})\,c^{(1)} + (b^{(1)} + b^{(5)})\,c^{(2)}]\cos 3\theta \\
&+ [\qquad\quad + (b^{(2)} - b^{(4)})\,s^{(1)} + (b^{(1)} - b^{(5)})\,s^{(2)}]\sin 3\theta \\
&+ \qquad\qquad \text{etc.} \qquad\qquad\qquad \text{etc.}
\end{aligned}\right\} \tag{31}$$

Now let

$$\left.\begin{aligned}
k_i\cos K_i &= b^{(i)}.c^{(0)} + (b^{(i-1)} + b^{(i+1)})\,c^{(1)} + (b^{(i-2)} + b^{(i+2)})\,c^{(2)} \\
k_i\sin K_i &= \qquad\qquad\; + (b^{(i-1)} - b^{(i+1)})\,s^{(1)} + (b^{(i-2)} - b^{(i+2)})\,s^{(2)}
\end{aligned}\right\} \tag{32}$$

and we find

$$\begin{aligned}
\left(\frac{a}{\varDelta}\right) &= k_i\,[\cos K_i.\cos i\theta + \sin K_i.\sin.i\theta] \\
&= k_i\cos(i\theta - K_i) = k_i.\cos(iQ - i\varepsilon' - K_i).
\end{aligned} \tag{33}$$

Subtracting and adding the angle ig, this becomes

$$\begin{aligned}
\left(\frac{a}{\varDelta}\right) &= k_i\cos\left[i(Q-g) - K_i + (ig - i\varepsilon')\right] \\
&= k_i\cos\left[i(Q-g) - K_i\right]\cos.i(g-\varepsilon') - k_i.\sin\left[i(Q-g) - K_i\right]\sin.i(g-\varepsilon')
\end{aligned} \tag{34}$$

If we put

$$\left.\begin{aligned}
\overset{(c)}{A}_{i,\kappa} &= \frac{2}{n}k_{i,\kappa}\cos\,[i(Q_\kappa - g_\kappa) - K_{i,\kappa}] \\
\overset{(s)}{A}_{i,\kappa} &= \frac{2}{n}k_{i,\kappa}\sin\,[i(Q_\kappa - g_\kappa) - K_{i,\kappa}],
\end{aligned}\right\} \tag{35}$$

n being the number of divisions, we find

$$\left(\frac{a}{\Delta}\right) = \overset{(c)}{A}_{i,\kappa} \,.\cos i\,(g_\kappa - \varepsilon'_\kappa) - \overset{(s)}{A}_{i,\kappa}\,.\sin i\,(g_\kappa - \varepsilon'_\kappa) \tag{36}$$

If now, for the purpose of multiplying the series together, we put

$$\left.\begin{aligned} \overset{(c)}{A}_{i,\kappa} &= \Sigma\, \overset{(c)}{C}_{i,\nu}\,.\cos \nu g + \Sigma\,.\, \overset{(s)}{C}_{i,\nu} \sin \nu g \\ \overset{(s)}{A}_{i,\kappa} &= \Sigma\, \overset{(c)}{S}_{i,\nu}\,.\cos \nu g + \Sigma\, \overset{(s)}{S}_{i,\nu}\,.\sin \nu g \end{aligned}\right\} \tag{37}$$

we have

$$\left(\frac{a}{\Delta}\right) = [\Sigma\, \overset{(c)}{C}_{i,\nu} \cos \nu g + \Sigma\, \overset{(s)}{C}_{i,\nu} \sin \nu g] \cos i\,(g-\varepsilon') - [\Sigma\, \overset{(c)}{S}_{i,\nu} \cos \nu g + \Sigma\, \overset{(s)}{S}_{i,\nu} \sin \nu g] \sin i\,(g-\varepsilon') \tag{38}$$

Performing the operations indicated we get

$$\begin{aligned}
\Sigma\Sigma \cos(ig - i\varepsilon').\overset{(c)}{C}_{i,\nu} \cos \nu g &= \Sigma\Sigma \tfrac{1}{2} \overset{(c)}{C}_{i,\nu} \cos[(i+\nu)g - i\varepsilon'] + \Sigma\Sigma \tfrac{1}{2} \overset{(c)}{C}_{i,\nu} \cos[(i-\nu)g - i\varepsilon'] \\
\Sigma\Sigma \cos(ig - i\varepsilon').\overset{(s)}{C}_{i,\nu} \sin \nu g &= \Sigma\Sigma \tfrac{1}{2} \overset{(s)}{C}_{i,\nu} \sin[(i+\nu)g - i\varepsilon'] - \Sigma\Sigma \tfrac{1}{2} \overset{(s)}{C}_{i,\nu} \sin[(i-\nu)g - i\varepsilon'] \\
-\Sigma\Sigma \sin(ig - i\varepsilon')\, \overset{(c)}{S}_{i,\nu} \cos \nu g &= -\Sigma\Sigma \tfrac{1}{2} \overset{(c)}{S}_{i,\nu} \sin[(i+\nu)g - i\varepsilon'] - \Sigma\Sigma \tfrac{1}{2} \overset{(c)}{S}_{i,\nu} \sin[(i-\nu)g - i\varepsilon'] \\
-\Sigma\Sigma \sin(ig - i\varepsilon')\, \overset{(s)}{S}_{i,\nu} \sin \nu g &= \Sigma\Sigma \tfrac{1}{2} \overset{(s)}{S}_{i,\nu} \cos[(i+\nu)g - i\varepsilon'] - \Sigma\Sigma \tfrac{1}{2} \overset{(s)}{S}_{i,\nu} \cos[(i-\nu)g - i\varepsilon']
\end{aligned}$$

Summing the terms we find

$$\left(\frac{a}{\Delta}\right)^n = \Sigma\Sigma \tfrac{1}{2} (\overset{(c)}{C}_{i,\nu} \mp \overset{(s)}{S}_{i,\nu}) \cos\left[(i \mp \nu)g - i\varepsilon'\right] \mp \tfrac{1}{2} \Sigma\Sigma (\overset{(s)}{C}_{i,\nu} \pm \overset{(c)}{S}_{i,\nu}) \sin\left[(i \mp \nu) g - i\varepsilon'\right] \tag{39}$$

From the formula of mechanical quadrature just given, we have $\overset{(c)}{C}_{i,0}$, $\overset{(c)}{S}_{i,0}$, when $\nu = 0$; but we know that they are $\frac{1}{2}\,.\,\overset{(c)}{C}_{i,0}$, $\frac{1}{2}\,\overset{(c)}{S}_{i,0}$, as shown by their derivation.

Thus

$$\left.\begin{aligned} \overset{(c)}{A}_i = \tfrac{1}{2} \overset{(c)}{C}_{i,0} + \overset{(c)}{C}_{i,1} \cos g + \overset{(c)}{C}_{i,2}\,.\cos 2g + \text{etc.} \\ + \overset{(s)}{C}_{i,1} \sin g + \overset{(s)}{C}_{i,2}\,.\sin 2g + \text{etc.} \end{aligned}\right\} = \Sigma \overset{(c)}{C}_{i,\nu} \cos \nu g + \Sigma \overset{(s)}{C}_{i,\nu} \sin \nu g$$

$$\left.\begin{aligned} \overset{(s)}{A}_i = \tfrac{1}{2} \overset{(c)}{S}_{i,0} + \overset{(c)}{S}_{i,1} \cos g + \overset{(c)}{S}_{i,2} \cos 2g + \text{etc.} \\ + \overset{(s)}{S}_{i,1} \sin g + \overset{(s)}{S}_{i,2} \sin 2g + \text{etc.} \end{aligned}\right\} = \Sigma \overset{(c)}{S}_{i,\nu} \cos \nu g + \Sigma \overset{(s)}{S}_{i,\nu} \sin \nu g.$$

Hence where $\nu = 0$, each series is reduced to its first term.

In the application of the very general formulæ care must be taken to note the signification of the various terms employed.

In case of

$$\overset{(c)}{A}_{i,\kappa} = \frac{2}{n}\, k_{i,\kappa} \,.\, \cos\, [i\,(Q_\kappa - g_\kappa) - K_{i,\kappa}]$$

$$\overset{(s)}{A}_{i,\kappa} = \frac{2}{n}\, k_{i,\kappa} \,.\, \sin\, [i\,(Q_\kappa - g_\kappa) - K_{i,\kappa}],$$

n shows the number of divisions of the circumference; and we divide by $\frac{n}{2}$ in forming $k_{i,\kappa}$ to save division when forming the coefficients c_ν, s_ν.

The index and multiple i shows the term in the series

$$\tfrac{1}{2}b^{(0)} + b^{(1)} \cos\, (\varepsilon' - Q) + b^{(2)} \,.\, \cos\, 2(\varepsilon' - Q) + b^{(3)} \,.\, \cos\, 3(\varepsilon' - Q) + \text{etc.}$$

The double index i, κ shows the term of the series of La Place's coefficients and the particular point in the circumference.

The index ν shows the general term of the series expressing the values of $\overset{(c)}{A}_{i,\kappa}$, $\overset{(s)}{A}_{i,\kappa}$, when we give to ν values from $\nu = 0$, to the highest value of ν needed in the approximation.

In $\frac{2}{n} \,.\, k_{i,\kappa}$, $i(Q_\kappa - g_\kappa) - K_{i,\kappa}$, for each value of i, there are n values of each quantity.

The next step is to express the n values of $\overset{(c)}{A}_0$, $\overset{(c)}{A}_1$, $\overset{(s)}{A}_1$, $\overset{(c)}{A}_2$, $\overset{(s)}{A}_2$, etc., respectively in terms of a periodic series. And since these quantities are functions of the mean anomaly g, if we designate them generally by Y, of which the special values are

$$Y_0 \quad , \quad Y_1 \quad , \quad Y_2 \quad , \; . \; . \; . \; . \; Y_{n-1} \quad ,$$

we have

$$\left.\begin{aligned} Y = \tfrac{1}{2}c_0 + c_1 \cos g + c_2 \cos 2g + \text{etc.} \\ + s_1 \sin g + s_2 \sin 2g + \text{etc.} \end{aligned}\right\} \qquad (40)$$

The values of c_ν, s_ν, in this series are found from the n special values of Y.

From

$$\overset{(c)}{A_1}, \text{ or } \overset{(s)}{A_1} = \tfrac{1}{2} c_0 + c_1 \cos g + c_2 \cos 2g + \text{etc.}$$
$$+ s_1 \sin g + s_2 \sin 2g + \text{etc.},$$

and similarly, for every other value of $\varkappa$ in $\overset{(c)}{A_{i,\kappa}}$, $\overset{(s)}{A_{i,\kappa}}$, we have a check on the values of c_ν, s_ν, in each series. Thus if in case of sixteen divisions of the circumference we take $g = 22.^\circ 5$ and find the value of the series, the sum of the terms must equal the value of $\overset{(c)}{A_{i,\kappa}}$, $\overset{(s)}{A_{i,\kappa}}$, corresponding to $g = 22.^\circ 5$. And this check should be employed on each series, using that value of g that gives the most values of c_ν and s_ν. If i extends to $i = 9$, we have ten separate checks for the values of $\overset{(c)}{A_{i,\kappa}}$, $\overset{(s)}{A_{i,\kappa}}$, respectively.

In the equation

$$Y = \tfrac{1}{2}c_0 + c_1 \,.\, \cos g + c_2 \,.\, \cos 2g + c_3 \,.\, \cos 3g + \text{etc.}$$
$$+ s_1 \,.\, \sin g + s_2 \,.\, \sin 2g + s_3 \,.\, \sin 3g + \text{etc.},$$

if the circumference is divided into twelve parts, each division is 30°. Then for the special values of Y we have

$$Y_0 = \tfrac{1}{2}c_0 + c_1 + c_2 + c_3 + \text{etc.}$$

$$Y_1 = \tfrac{1}{2}c_0 + c_1 \,.\, \cos 30^\circ + c_2 \,.\, \cos 60^\circ + c_3 \cos 90^\circ + \text{etc.}$$
$$+ s_1 \sin 30^\circ + s_2 \sin 60^\circ + s_3 \sin 90^\circ + \text{etc.}$$

$$Y_2 = \tfrac{1}{2}c_0 + c_1 \,.\, \cos 60^\circ + c_2 \,.\, \cos 120^\circ + c_3 \cos 180^\circ + \text{etc.}$$
$$+ s_1 \sin 60^\circ + s_2 \,.\, \sin 120^\circ + s_3 \sin 180^\circ + \text{etc.}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$Y_{11} = \tfrac{1}{2}c_0 + c_1 \,.\, 330^\circ + c_2 \,.\, \cos 300^\circ + c_3 \cos 270^\circ + \text{etc.}$$
$$+ s_1 \,.\, 330^\circ + s_2 \,.\, \sin 300^\circ + s_3 \sin 270^\circ + \text{etc.}$$

In the same way we proceed for any other number of divisions of the circumference.

Now let

$$
\begin{aligned}
(0.6\ ) &= Y_0 + Y_6 & \left(\tfrac{0}{6}\right) &= Y_0 - Y_6 \\
(1.7\ ) &= Y_1 + Y_7 & \left(\tfrac{1}{7}\right) &= Y_1 - Y_7 \\
(2.8\ ) &= Y_2 + Y_8 & \left(\tfrac{2}{8}\right) &= Y_2 - Y_8 \\
&\vdots & &\vdots \\
(5.11) &= Y_5 + Y_{11} & \left(\tfrac{5}{11}\right) &= Y_5 - Y_{11}
\end{aligned}
$$

Then

$$
\begin{aligned}
3(c_0 + 2c_6) &= (0.6) + (2.8) + (4.10) \\
3(c_0 - 2c_6) &= (1.7) + (3.9) + (5.11) \\
3(c_2 + c_4) &= (0.6) - \left[(2.8) + (4.10)\right] \sin 30^\circ \\
3(c_2 - c_4) &= \left[(1.7) + (5.11)\right] \sin 30^\circ - (3.9) \\
3(s_2 + s_4) &= \left[(1.7) - (5.11)\right] \cos 30^\circ \\
3(s_2 - s_4) &= \left[(2.8) - (4.10)\right] \cos 30^\circ \\
3(c_1 + c_5) &= \left(\tfrac{0}{6}\right) + \left[\left(\tfrac{2}{8}\right) - \left(\tfrac{4}{10}\right)\right] \sin 30^\circ \\
3(c_1 - c_5) &= \left[\left(\tfrac{1}{7}\right) - \left(\tfrac{5}{11}\right)\right] \cos 30^\circ \\
6 . c_3 &= \left(\tfrac{0}{6}\right) - \left(\tfrac{2}{8}\right) + \left(\tfrac{4}{10}\right) \\
3(s_1 + s_5) &= \left[\left(\tfrac{1}{7}\right) + \left(\tfrac{5}{11}\right)\right] \sin 30^\circ + \left(\tfrac{3}{9}\right) \\
3(s_1 - s_5) &= \left[\left(\tfrac{2}{8}\right) + \left(\tfrac{4}{10}\right)\right] \cos 30^\circ \\
6 . s_3 &= \left(\tfrac{1}{7}\right) - \left(\tfrac{3}{9}\right) + \left(\tfrac{5}{11}\right).
\end{aligned}
$$

The values of these coefficients can be easily verified by finding the values of each one from the sum for all the different values of Y as given in the series for $Y_0, Y_1, Y_2, \ldots . Y_{11}$.

When we divide the circumference into sixteen parts, each division is 22.° 5. We find the values of $Y_0, Y_1, Y_2, \ldots . Y_{15}$, as in the case of twelve divisions. To find the values of c_ν and s_ν, in the case of sixteen divisions, we put

$$
\begin{aligned}
(0.8\ ) &= Y_0 + Y_8 & \left(\tfrac{0}{8}\right) &= Y_0 - Y_8 \\
(1.9\ ) &= Y_1 + Y_9 & \left(\tfrac{1}{9}\right) &= Y_1 - Y_9 \\
(2.10) &= Y_2 + Y_{10} & \left(\tfrac{2}{10}\right) &= Y_2 - Y_{10} \\
\vdots \quad &\quad \vdots & \vdots \quad &\quad \vdots \\
(7.15) &= Y_7 + Y_{15} & \left(\tfrac{7}{15}\right) &= Y_7 - Y_{15}
\end{aligned}
$$

$$
\begin{aligned}
(0.4) &= (0.8) + (4.12) \qquad & (0.2) &= (0.4) + (2.6)\\
(1.5) &= (1.9) + (5.13) \qquad & (1.3) &= (1.5) + (3.7)\\
(2.6) &= (2.10) + (6.14) & &\\
(3.7) &= (3.11) + (7.15). & &
\end{aligned}
$$

Then

$$
\begin{aligned}
4(c_0 + 2.c_8) &= (0.2)\\
4(c_0 - 2.c_8) &= (1.3)\\
4(c_2 + c_6) &= (0.8) - (4.12)\\
4(c_2 - c_6) &= \{[(1.9) - (5.13)] - [(3.11) - (7.15)]\} \cos 45^\circ\\
4(s_2 + s_6) &= \{[(1.9) - (5.13)] + [(3.11) - (7.15)]\} \cos 45^\circ\\
4(s_2 - s_6) &= (2.10) - (6.14)\\
8.c_4 &= (0.4) - (2.6)\\
8.s_4 &= (1.5) - (3.7)\\
4(c_1 + c_7) &= (\tfrac{0}{8}) + [(\tfrac{2}{10}) - (\tfrac{6}{14})] \cos 45^\circ\\
4(c_1 - c_7) &= [(\tfrac{1}{9}) - (\tfrac{7}{15})] \cos 22.^\circ 5 + [(\tfrac{3}{11}) - (\tfrac{5}{13})] \cos 67.^\circ 5\\
4(c_3 + c_5) &= (\tfrac{0}{8}) - [(\tfrac{2}{10}) - (\tfrac{6}{14})] \cos 45^\circ\\
4(c_3 - c_5) &= [(\tfrac{1}{9}) - (\tfrac{7}{15})] \sin 22.^\circ 5 - [(\tfrac{3}{11}) - (\tfrac{5}{13})] \sin 67.^\circ 5\\
4(s_1 + s_7) &= [(\tfrac{1}{9}) + (\tfrac{7}{15})] \sin 22.^\circ 5 + [(\tfrac{3}{11}) + (\tfrac{5}{13})] \sin 67.^\circ 5\\
4(s_1 - s_7) &= [(\tfrac{2}{10}) + (\tfrac{6}{14})] \cos 45^\circ + (\tfrac{4}{12})\\
4(s_3 + s_5) &= [(\tfrac{1}{9}) + (\tfrac{7}{15})] \cos 22.^\circ 5 - [(\tfrac{3}{11}) + (\tfrac{5}{13})] \cos 67.^\circ 5\\
4(s_3 - s_5) &= [(\tfrac{2}{10}) + (\tfrac{6}{14})] \cos 45^\circ - (\tfrac{4}{12}).
\end{aligned}
$$

When the circumference is divided into twenty-four parts, each part is 15°.

Let

$$
\begin{array}{lll}
(0.12) = Y_0 + Y_{12} & (0.6) = (0.12) + (6.18) & (\tfrac{0}{6}) = (0.12) - (6.18)\\
(1.13) = Y_1 + Y_{13} & (1.7) = (1.13) + (7.19) & (\tfrac{1}{7}) = (1.13) - (7.19)\\
(2.14) = Y_2 + Y_{14} & (2.8) = (2.14) + (8.20) & (\tfrac{2}{8}) = (2.14) - (8.20)\\
\vdots \qquad \vdots & \vdots \qquad \vdots & \vdots \qquad \vdots\\
(11.23) = Y_{11} + Y_{23} & (5.11) = (5.17) + (11.23) & (\tfrac{5}{11}) = (5.17) - (11.23)
\end{array}
$$

Then

$$
\begin{aligned}
6(c_0+2\,.\,c_{12}) &= (0\,.\,6)+(2\,.\,8)+(4\,.\,10)\\
6(c_0-2\,.\,c_{12}) &= (1\,.\,7)+(3\,.\,9)+(5\,.\,11)\\
6(c_2+c_{10}) &= (\tfrac{0}{6})+\left[(\tfrac{2}{8})-(\tfrac{4}{10})\right]\sin 30^\circ\\
6(c_2-c_{10}) &= \left[(\tfrac{1}{7})-(\tfrac{5}{11})\right]\cos 30^\circ\\
6(c_4+c_8) &= (0\,.\,6)-\left[(2\,.\,8)+(4\,.\,10)\right]\sin 30^\circ\\
6(c_4-c_8) &= \left[(1\,.\,7)+(5\,.\,11)\right]\sin 30^\circ-(3\,.\,9)\\
6)s_2+s_{10}) &= \left[(\tfrac{1}{7})+(\tfrac{5}{11})\right]\sin 30^\circ+(\tfrac{3}{9})\\
6(s_2-s_{10}) &= \left[(\tfrac{2}{8})+(\tfrac{4}{10})\right]\cos 30^\circ\\
6(s_4+s_8) &= \left[(\tfrac{1}{7})-(\tfrac{5}{11})\right]\cos 30^\circ\\
6(s_4-s_8) &= \left[(\tfrac{2}{8})-(\tfrac{4}{10})\right]\cos 30^\circ\\
12\,.\,c_6 &= (\tfrac{0}{6})-(\tfrac{2}{8})+(\tfrac{4}{10})\\
12\,.\,s_6 &= (\tfrac{1}{7})-(\tfrac{3}{9})+(\tfrac{5}{11})
\end{aligned}
$$

Further, let

$$
\begin{aligned}
(\tfrac{0}{12}) &= Y_0 - Y_{12}\\
(\tfrac{1}{13}) &= Y_1 - Y_{13}\\
(\tfrac{2}{14}) &= Y_2 - Y_{14}\\
&\vdots\\
(\tfrac{11}{23}) &= Y_{11} - Y_{23}
\end{aligned}
$$

Then

$$
\begin{aligned}
6(c_1+c_{11}) &= (\tfrac{0}{12})+\left[(\tfrac{2}{14})-(\tfrac{10}{22})\right]\cos 30^\circ+\left[(\tfrac{4}{16})-(\tfrac{8}{20})\right]\cos 60^\circ\\
6(c_1-c_{11}) &= \left[(\tfrac{1}{13})-(\tfrac{11}{23})\right]\cos 15^\circ+\left[(\tfrac{3}{15})-(\tfrac{9}{21})\right]\cos 45^\circ+\left[(\tfrac{5}{17})-(\tfrac{7}{19})\right]\cos 75^\circ\\
6(c_3+c_9) &= (\tfrac{0}{12})-(\tfrac{4}{16})+(\tfrac{8}{20})\\
6(c_3-c_9) &= \left\{(\tfrac{1}{13})-(\tfrac{11}{23})-\left[(\tfrac{3}{15})-(\tfrac{9}{21})\right]-\left[(\tfrac{5}{17})-(\tfrac{7}{19})\right]\right\}\cos 45^\circ\\
6(c_5+c_7) &= (\tfrac{0}{12})-\left[(\tfrac{2}{14})-(\tfrac{10}{22})\right]\cos 30^\circ+\left[(\tfrac{4}{16})-(\tfrac{8}{20})\right]\cos 60^\circ\\
6(c_5-c_7) &= \left[(\tfrac{1}{13})-(\tfrac{11}{23})\right]\sin 15^\circ-\left[(\tfrac{3}{15})-(\tfrac{9}{21})\right]\sin 45^\circ+\left[(\tfrac{5}{17})-(\tfrac{7}{19})\right]\sin 75^\circ\\
6(s_1+s_{11}) &= \left[(\tfrac{1}{13})+(\tfrac{11}{23})\right]\sin 15^\circ+\left[(\tfrac{3}{15})+(\tfrac{9}{21})\right]\sin 45^\circ+\left[(\tfrac{5}{17})+(\tfrac{7}{19})\right]\sin 75^\circ\\
6(s_1-s_{11}) &= \left[(\tfrac{2}{14})+(\tfrac{10}{22})\right]\sin 30^\circ+\left[(\tfrac{4}{16})+(\tfrac{8}{20})\right]\sin 60^\circ+(\tfrac{6}{18})\\
6(s_3+s_9) &= \left\{(\tfrac{1}{13})+(\tfrac{11}{23})+(\tfrac{3}{15})+(\tfrac{9}{21})-\left[(\tfrac{5}{17})+(\tfrac{7}{19})\right]\right\}\cos 45^\circ\\
6(s_3-s_9) &= (\tfrac{2}{14})-(\tfrac{6}{18})+(\tfrac{10}{22})\\
6(s_5+s_7) &= \left[(\tfrac{1}{13})+(\tfrac{11}{23})\right]\cos 15^\circ-\left[(\tfrac{3}{15})+(\tfrac{9}{21})\right]\cos 45^\circ+\left[(\tfrac{5}{17})+(\tfrac{7}{19})\right]\cos 75^\circ\\
6(s_5-s_7) &= \left[(\tfrac{2}{14})+(\tfrac{10}{22})\right]\sin 30^\circ-\left[(\tfrac{4}{16})+(\tfrac{8}{20})\right]\sin 60^\circ+(\tfrac{6}{18}).
\end{aligned}
$$

When the circumference is divided into thirty-two parts, each part is $11^\circ.25$ Let

$$
\begin{array}{lll}
(\ 0.16) = Y_0 + Y_{16} & (0.8\ ) = (0.16) + (\ 8.24) & (0.4) = (0.8\ ) + (4.12) \\
(\ 1.17) = Y_1 + Y_{17} & (1.9\ ) = (1.17) + (\ 9.25) & (1.5) = (1.9\ ) + (5.13) \\
(\ 2.18) = Y_2 + Y_{18} & (2.10) = (2.18) + (10.26) & (2.6) = (2.10) + (6.14) \\
\vdots & \vdots & (3.7) = (3.11) + (7.15) \\
(15.31) = Y_{15} + Y_{31} & (7.15) = (7.23) + (15.31) & (0.2) = (0.4\ ) + (2.6\ ) \\
 & & (1.3) = (1.5\ ) + (3.7\ )
\end{array}
$$

$$
\begin{array}{ll}
(\tfrac{0}{8}) = (0.16) - (\ 8.24) & (\tfrac{0}{4}) = (0.8\ ) - (4.12) \\
(\tfrac{1}{9}) = (1.17) - (\ 9.25) & (\tfrac{1}{5}) = (1.9\ ) - (5.13) \\
\vdots & (\tfrac{2}{6}) = (2.10) - (6.14) \\
(\tfrac{7}{15}) = (7.23) - (15.31) & (\tfrac{3}{7}) = (3.11) - (7.15)
\end{array}
$$

Then

$$
\begin{aligned}
8(c_0 + 2.c_{16}) &= (0.2) + (1.3) \\
8(c_0 - 2.c_{16}) &= (0.2) - (1.3) \\
8(c_2 + c_{14}) &= (\tfrac{0}{8}) + \left[(\tfrac{2}{10}) - (\tfrac{6}{14})\right] \cos 45^\circ \\
8(c_2 - c_{14}) &= \left[(\tfrac{1}{9}) - (\tfrac{7}{15})\right] \cos 22.^\circ 5 + \left[(\tfrac{3}{11}) - (\tfrac{5}{13})\right] \cos 67.^\circ 5 \\
8(c_4 + c_{12}) &= (\tfrac{0}{4}) \\
8(c_4 - c_{12}) &= \left[(\tfrac{1}{5}) - (\tfrac{3}{7})\right] \cos 45^\circ \\
8(c_6 + c_{10}) &= (\tfrac{0}{8}) - \left[(\tfrac{2}{10}) - (\tfrac{6}{14})\right] \cos 45^\circ \\
8(c_6 - c_{10}) &= \left[(\tfrac{1}{9}) - (\tfrac{7}{15})\right] \sin 22.^\circ 5 - \left[(\tfrac{3}{11}) - (\tfrac{5}{13})\right] \sin 67.^\circ 5 \\
16.c_8 &= (0.4) - (2.6) \\
8(s_2 + s_{14}) &= \left[(\tfrac{1}{9}) + (\tfrac{7}{15})\right] \sin 22.^\circ 5 + \left[(\tfrac{3}{11}) + (\tfrac{5}{13})\right] \sin 67.^\circ 5 \\
8(s_2 - s_{14}) &= \left[(\tfrac{2}{10}) - (\tfrac{6}{14})\right] \cos 45^\circ + (\tfrac{4}{12}) \\
8(s_4 + s_{12}) &= \left[(\tfrac{1}{5}) + (\tfrac{3}{7})\right] \cos 45^\circ \\
8(s_4 - s_{12}) &= (\tfrac{2}{6}) \\
8(s_6 + s_{10}) &= \left[(\tfrac{1}{9}) + (\tfrac{7}{15})\right] \cos 22.^\circ 5 - \left[(\tfrac{3}{11}) + (\tfrac{5}{13})\right] \cos 67.^\circ 5 \\
8(s_6 - s_{10}) &= \left[(\tfrac{2}{10}) - (\tfrac{6}{14})\right] \cos 45^\circ - (\tfrac{4}{12}).
\end{aligned}
$$

Further, let

$$
\begin{aligned}
(\tfrac{0}{16}) &= Y_0 - Y_{16} \\
(\tfrac{1}{17}) &= Y_1 - Y_{17} \\
(\tfrac{2}{18}) &= Y_2 - Y_{18} \\
&\vdots \\
(\tfrac{15}{31}) &= Y_{15} - Y_{31}
\end{aligned}
$$

And besides, let

$$
\begin{aligned}
A &= \left[(\tfrac{1}{17}) - (\tfrac{15}{31})\right] \cos 11^\circ.25 + \left[(\tfrac{7}{23}) - (\tfrac{9}{25})\right] \cos 78^\circ.75 \\
B &= \left[(\tfrac{1}{17}) - (\tfrac{15}{31})\right] \sin 11^\circ.25 - \left[(\tfrac{7}{23}) - (\tfrac{9}{25})\right] \sin 78^\circ.75 \\
A' &= \left[(\tfrac{2}{18}) - (\tfrac{14}{30})\right] \cos 22^\circ.5 + \left[(\tfrac{6}{22}) - (\tfrac{10}{26})\right] \cos 67^\circ.5 \\
B' &= \left[(\tfrac{2}{18}) - (\tfrac{14}{30})\right] \sin 22^\circ.5 - \left[(\tfrac{6}{22}) - (\tfrac{10}{26})\right] \sin 67^\circ.5 \\
A'' &= \left[(\tfrac{3}{19}) - (\tfrac{13}{29})\right] \cos 33^\circ.75 + \left[(\tfrac{5}{21}) - (\tfrac{11}{27})\right] \cos 56^\circ.25 \\
B'' &= \left[(\tfrac{3}{19}) - (\tfrac{13}{29})\right] \sin 33^\circ.75 - \left[(\tfrac{5}{21}) - (\tfrac{11}{27})\right] \sin 56^\circ.25 \\
A''' &= (\tfrac{0}{16}) + \left[(\tfrac{4}{20}) - (\tfrac{12}{28})\right] \cos 45^\circ \\
B''' &= (\tfrac{0}{16}) - \left[(\tfrac{4}{20}) - (\tfrac{12}{28})\right] \cos 45^\circ \\
C &= \left[(\tfrac{1}{17}) + (\tfrac{15}{31})\right] \sin 11^\circ.25 + \left[(\tfrac{7}{23}) + (\tfrac{9}{25})\right] \sin 78^\circ.75 \\
D &= \left[(\tfrac{1}{17}) + (\tfrac{15}{31})\right] \cos 11^\circ.25 - \left[(\tfrac{7}{23}) + (\tfrac{9}{25})\right] \cos 78^\circ.75 \\
C' &= \left[(\tfrac{2}{18}) + (\tfrac{14}{30})\right] \sin 22^\circ.5 + \left[(\tfrac{6}{22}) + (\tfrac{10}{26})\right] \sin 67^\circ.5 \\
D' &= \left[(\tfrac{2}{18}) + (\tfrac{14}{30})\right] \cos 22^\circ.5 - \left[(\tfrac{6}{22}) + (\tfrac{10}{26})\right] \cos 67^\circ.5 \\
C'' &= \left[(\tfrac{3}{19}) + (\tfrac{13}{29})\right] \sin 33^\circ.75 + \left[(\tfrac{5}{21}) + (\tfrac{11}{27})\right] \sin 56^\circ.25 \\
D'' &= \left[(\tfrac{3}{19}) + (\tfrac{13}{29})\right] \cos 33^\circ.75 - \left[(\tfrac{5}{21}) + (\tfrac{11}{27})\right] \cos 56^\circ.25 \\
C''' &= \left[(\tfrac{4}{20}) + (\tfrac{12}{28})\right] \cos 45^\circ + (\tfrac{8}{24}) \\
D''' &= \left[(\tfrac{4}{20}) + (\tfrac{12}{28})\right] \cos 45^\circ - (\tfrac{8}{24}).
\end{aligned}
$$

Then

$$8(c_1+c_{15}) = A''' + A'$$
$$8(c_1-c_{15}) = A + A''$$
$$8(c_3+c_{13}) = B''' + B'$$
$$8(c_3-c_{13}) = [A - A'' + B + B''] \cos 45^\circ$$
$$8(c_5+c_{11}) = B''' - B'$$
$$8(c_5-c_{11}) = \left[A - A'' - (B + B'')\right] \cos 45^\circ$$
$$8(c_7+c_9) = A''' - A'$$
$$8(c_7-c_9) = B - B''$$

$$8(s_1+s_{15}) = C + C''$$
$$8(s_1-s_{15}) = C''' + C'$$
$$8(s_3+s_{13}) = \left[D + D'' - (C - C'')\right] \cos 45^\circ$$
$$8(s_3-s_{13}) = D' + D'''$$
$$8(s_5+s_{11}) = [D + D'' + C - C''] \cos 45^\circ$$
$$8(s_5-s_{11}) = D' - D'''$$
$$8(s_7+s_9) = D - D''$$
$$8(s_7-s_9) = -C''' + C'.$$

The expressions for the determination of the values of c_ν and s_ν, just given, are found in HANSEN'S *Auseinandersetzung*, Band I, Seite 159–164.

CHAPTER II.

Derivation of the Expressions for BESSEL'S *Functions for the Transformation of Trigonometric Series.*

The value of $\left(\frac{a}{\varDelta}\right)^n$ given thus far is found expressed in a series of terms the arguments of which have the eccentric anomaly of the disturbing body as one constituent. But as the mean anomaly of both bodies is to be employed, it will be necessary to make one transformation; and the next step will be to develop the necessary formulæ for this purpose. HANSEN, in his work entitled *Entwickelung des Products einer Potenz des Radius Vectors et cet.*, has treated the subject of transforming from one anomaly into another very fully; what is here given is based mainly on this work.

Calling c the Naperian base, and putting

$$y = c^{\varepsilon\sqrt{-1}}, \qquad y' = c^{\varepsilon'\sqrt{-1}},$$

we have

$$y\,y' = (\cos\varepsilon + \sqrt{-1}\sin\varepsilon)\,(\cos\varepsilon' + \sqrt{-1}\sin\varepsilon');$$

also

$$\begin{aligned} y^i\,y'^{i'} &= (\cos i\,\varepsilon + \sqrt{-1}\sin i\,\varepsilon')\,(\cos i'\,\varepsilon' + \sqrt{-1}\sin i'\,\varepsilon') \\ &= \cos(i\,\varepsilon - i'\,\varepsilon') + \sqrt{-1}\sin(i\,\varepsilon' - i'\,\varepsilon'). \end{aligned}$$

Denoting the cosine and sine coefficients of the angles $(i\,\varepsilon - i'\,\varepsilon')$ by (i, i', c) and (i, i', s) respectively, the series

$$F = \Sigma\Sigma\,(i, i', c)\cos(i\,\varepsilon - i'\,\varepsilon') - \Sigma\Sigma\sqrt{-1}\,(i, i', s)\sin(i\,\varepsilon - i'\varepsilon') \tag{1}$$

can be put in the form

$$F = \tfrac{1}{2}\Sigma\Sigma\,\left\{(i, i', c) - \sqrt{-1}\,(i, i', s)\right\}\,y^i\,y'^{i'}. \tag{2}$$

In a similar manner we get

$$F = \tfrac{1}{2}.\Sigma\Sigma \left\{((i,h',c)) - \sqrt{-1}\,((i,h',s))\,y^i . z'^{-h'}\right.,\tag{3}$$

where

$$z' = c^{-g'\sqrt{-1}}.$$

We have now to find the relation between y and z.

Let

$$g = \text{the mean anomaly,}$$
$$\text{and } \varepsilon = \text{the eccentric anomaly.}$$

Then from

$$g = \varepsilon - e \sin \varepsilon,$$

introducing $\sqrt{-1}$, we get

$$g\sqrt{-1} = \varepsilon\sqrt{-1} - e \sin \varepsilon \sqrt{-1}.$$

Since

$$2\sqrt{-1}.\sin\varepsilon = y - y^{-1},$$

we find

$$g\sqrt{-1} = \varepsilon\sqrt{-1} - \frac{e}{2}(y - y^{-1}).$$

Now from

$$z = c^{g\sqrt{-1}},$$
$$y = c^{\varepsilon\sqrt{-1}},$$

we obtain

$$g\sqrt{-1} = \log. z,$$
$$\varepsilon\sqrt{-1} = \log. y,$$

and

$$\frac{e}{2}(y - y^{-1}) = \log.\left(c^{\frac{e}{2}(y - y^{-1})}\right). \tag{4}$$

Thus

$$g\sqrt{-1} = \log. z = \log.\left(y \,.\, c^{-\frac{e}{2}(y - y^{-1})}\right);$$

and hence

$$z = y \,.\, c^{-\frac{e}{2}(y - y^{-1})}. \tag{5}$$

From

$$z = y \,.\, c^{-\frac{e}{2}(y - y^{-1})},$$

we have

$$z^h = y^h \,.\, c^{-\frac{he}{2}(y - y^{-1})}, \tag{6}$$

and

$$y^i = z^i \,.\, c^{\frac{ie}{2}(y - y^{-1})}. \tag{7}$$

Let $\frac{e}{2}$ be denoted by λ; then

$$c^{-\frac{he}{2}(y - y^{-1})} = c^{-h\lambda \,.\, y} \,.\, c^{h\lambda \,.\, y^{-1}}, \tag{8}$$

and

$$c^{\frac{ie}{2}(y - y^{-1})} = c^{i\lambda \,.\, y} \,.\, c^{-i\lambda \,.\, y^{-1}}. \tag{9}$$

But

$$c^{-h\lambda.y} \,.\, c^{h\lambda\, y^{-1}} = \left(1 - h\lambda \,.\, y + \frac{h^2\lambda^2}{1.2} \cdot y^2 - \frac{h^3\lambda^3}{1.2.3} \cdot y^3 + \frac{h^4\lambda^4}{1.2.3.4} \cdot y^4 \mp \text{etc.}\right)$$
$$\left(1 + h\lambda \,.\, y^{-1} + \frac{h^2\lambda^2}{1.2} \cdot y^{-2} + \frac{h^3\lambda^3}{1.2.3} \cdot y^{-3} + \frac{h^4\lambda^4}{1.2\,3.4} \cdot y^{-4} + \text{etc.}\right)$$

and

$$c^{i\lambda.y} \cdot c^{-i\lambda y^{-1}} = \left(1 + i\lambda \cdot y + \frac{i^2.\lambda^2}{1.2} \cdot y^2 + \frac{i^3\lambda^3}{1.2.3} \cdot y^3 + \frac{i^4.\lambda^4}{1.2.3.4} \cdot y^4 + \text{etc.}\right)$$
$$\left(1 - i\lambda \cdot y^{-1} + \frac{i^2\lambda^2}{1.2} \cdot y^{-2} - \frac{i^3.\lambda^3}{1.2.3} \cdot y^{-3} + \frac{i^4\lambda^4}{1.2.3.4} \cdot y^{-4} + \text{etc.}\right)$$

Performing the operations indicated, we have

$$c^{-h\frac{e}{2}(y-y^{-1})} = \left(1 - h^2\lambda^2 + \frac{h^4\lambda^4}{1^2.2^2} - \frac{h^6.\lambda^6}{1^2.2^2.3^2} + \frac{h^8.\lambda^8}{1^2.2^2.3^2.4^2} \mp \text{etc.}\right)$$
$$\left(h\lambda - \frac{h^3\lambda^3}{1^2.2} + \frac{h^5.\lambda^5}{1^2.2^2.3} - \frac{h^7.\lambda^7}{1^2.2^2.3^2.4} \pm \text{etc.}\right)\left(y^{-1} - y\right)$$
$$\left(+ \frac{h^2\lambda^2}{1.2} - \frac{h^4\lambda^4}{1^2.2.3} + \frac{h^6\lambda^6}{1^2.2^2.3.4} \mp \text{etc.}\right)\left(y^{-2} + y^2\right)$$
$$\left(+ \frac{h^3\lambda^3}{1.2.3} - \frac{h^5\lambda^5}{1^2.2.3.4} \pm \text{etc.}\right)\left(y^{-3} - y^3\right)$$
$$\left(+ \frac{h^4\lambda^4}{1.2.3.4} \mp \text{etc.}\right)\left(y^{-4} + y^4\right);$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ \frac{h^m\lambda^m}{1.2..m}\left(1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4\lambda^4}{1.2.m+1.m+2} \mp \text{etc.}\right)y^m$$

$$c^{i\frac{e}{2}(y-y^{-1})} = 1 - i^2\lambda^2 + \frac{i^4\lambda^4}{1^2.2^2} - \frac{i^6\lambda^6}{1^2.2^2.3^2} + \frac{i^8\lambda^8}{1^2.2^2.3^2.4^2} \mp \text{etc.}$$
$$\left(+ i\lambda - \frac{i^3\lambda^3}{1^2.2} + \frac{i^5\lambda^5}{1^2.2^2.3} - \frac{i^7\lambda^7}{1^2.2^2.3^2.4} \pm \text{etc.}\right)\left(y - y^{-1}\right)$$
$$\left(+ \frac{i^2\lambda^2}{1.2} - \frac{i^4\lambda^4}{1^2.2.3} + \frac{i^6\lambda^6}{1^2.2^2.3.4} \mp \text{etc.}\right)\left(y^2 + y^{-2}\right)$$
$$\left(+ \frac{i^3\lambda^3}{1.2.3} - \frac{i^5\lambda^5}{1^2.2.3.4} \pm \text{etc.}\right)\left(y^3 - y^{-3}\right)$$
$$\left(+ \frac{i^4\lambda^4}{1.2.3.4} \mp \text{etc.}\right)\left(y^4 + y^{-4}\right)$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

As we may write h in place of i, we have, thus, also given the value of $c^{h\frac{e}{2}(y-y^{-1})}$.

Now put

$$\left.\begin{aligned} c^{-h\frac{e}{2}(y-y^{-1})} &= \Sigma_{-\infty}^{+\infty} \overset{(-m)}{J}_{-h\lambda} \,.\, y^{-m}, \\ c^{h\frac{e}{2}(y-y^{-1})} &= \Sigma_{-\infty}^{+\infty} \overset{(m)}{J}_{-h\lambda} \,.\, y^{m}. \end{aligned}\right\} \tag{10}$$

Then, from the preceding developments, we see that

$$\left.\begin{aligned} \overset{(-m)}{J}_{h\lambda} &= (-1)^m \,.\, \overset{(m)}{J}_{h\lambda}, \\ \overset{(m)}{J}_{-h\lambda} &= (-1)^m \,.\, \overset{(m)}{J}_{h\lambda}, \\ \overset{(-m)}{J}_{-h\lambda} &= \overset{(m)}{J}_{h\lambda}. \end{aligned}\right\} \tag{11}$$

Again

$$\left.\begin{aligned} \Sigma_{-\infty}^{+\infty} \overset{(-m)}{J}_{-h\lambda} \,.\, y^{-m} = {} & \overset{(0)}{J}_{-h\lambda} + \overset{(-1)}{J}_{-h\lambda} \,.\, y^{-1} + \overset{(-2)}{J}_{-h\lambda} \,.\, y^{-2} + \overset{(-3)}{J}_{-h\lambda} \,.\, y^{-3} + \text{etc.} \\ & + \overset{(1)}{J}_{-h\lambda} \,.\, y + \overset{(2)}{J}_{-h\lambda} \,.\, y^{2} + \overset{(3)}{J}_{-h\lambda} \,.\, y^{3} + \text{etc.} \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} \Sigma_{-\infty}^{+\infty} \overset{(m)}{J}_{h\lambda} \,.\, y^{m} = {} & \overset{(0)}{J}_{h\lambda} + \overset{(1)}{J}_{h\lambda} \,.\, y + \overset{(2)}{J}_{h\lambda} \,.\, y^{2} + \overset{(3)}{J}_{h\lambda} \,.\, y^{3} + \text{etc.} \\ & + \overset{(-1)}{J}_{h\lambda} \,.\, y^{-1} + \overset{(-2)}{J}_{h\lambda} \,.\, y^{-2} + \overset{(-3)}{J}_{h\lambda} \,.\, y^{-3} + \text{etc.} \end{aligned}\right\} \tag{13}$$

Comparing the values of $\Sigma_{-\infty}^{+\infty} \overset{(-m)}{J}_{-h\lambda} \,.\, y^{-m}$ and $c^{-h\frac{e}{2}(y-y^{-1})}$

we have

$$\begin{aligned} \overset{(-1)}{J}_{-h\lambda} = \overset{(1)}{J}_{h\lambda} &= h\lambda - \frac{h^3\lambda^3}{1^2.2} + \frac{h^5\lambda^5}{1^2.2^2.3} - \frac{h^7\lambda^7}{1^2.2^2.3^2.4} \pm \text{etc.}, && \text{for } y^{-1}, \\ -\overset{(1)}{J}_{-h\lambda} = \overset{(1)}{J}_{h\lambda} &= h\lambda - \frac{h^3\lambda^3}{1^2.2} + \frac{h^5\lambda^5}{1^2.2^2.3} - \frac{h^7\lambda^7}{1^2.2^2.3^2.4} \pm \text{etc.}, && \text{for } y^{1}, \\ \overset{(-2)}{J}_{-h\lambda} = \overset{(2)}{J}_{h\lambda} &= \frac{h^2\lambda^2}{1.2} - \frac{h^4\lambda^4}{1^2.2.3} + \frac{h^6\lambda^6}{1^2.2^2.3.4} \mp \text{etc.}, && \text{for } y^{-2}, \\ \overset{(2)}{J}_{-h\lambda} = \overset{(2)}{J}_{h\lambda} &= \frac{h^2\lambda^2}{1.2} - \frac{h^4\lambda^4}{1^2.2.3} + \frac{h^6\lambda^6}{1^2.2^2.3.4} \mp \text{etc.}, && \text{for } y^{2}, \\ \text{etc.} = \text{etc.} &= \text{etc.} \end{aligned}$$

Comparing the values of $\Sigma_{-\infty}^{+\infty} \overset{(m)}{J_{h\lambda}} . y^m$ and $c^{h\frac{e}{2}(y-y^{-1})}$, we get the same expressions for y^m and y^{-m}.

We see from the values of $\overset{(1)}{J_{h\lambda}}$, $\overset{(2)}{J_{h\lambda}}$, etc., found above, that the general term is

$$\overset{(m)}{J_{h\lambda}} = \frac{h^m\lambda^m}{1.2...m} - \frac{h^{m+2}.\lambda^{m+2}}{1^2.2..m.m+1} + \frac{h^{m+4}\,\lambda^{m+4}}{1^2.2^2...m.m+1.m+2} \mp \text{etc.}$$

$$= \frac{h^m\lambda^m}{1.2....m}\left(1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4\lambda^4}{1.2.m+1.m+2} \mp \text{etc.}\right) \tag{14}$$

Further, we have

$$z^h = c^{-h\frac{e}{2}(y-y^{-1})} . y^h \qquad = \overset{(m)}{J_{h\lambda}} . y^{-m} . y^h ;$$

and, by putting $m = h - i$, this becomes

$$z^h = \overset{(h-i)}{J_{h\lambda}} . y^i \tag{15}$$

Let

$$\left.\begin{aligned} z^h &= \Sigma_{-\infty}^{+\infty} \overset{(h)}{Q_i} . y^i \\ y^i &= \Sigma_{-\infty}^{+\infty} \overset{(i)}{P_h} . z^h \end{aligned}\right\} \tag{16}$$

Multiplying the second of these equations by $z^{-h} . dg$, we obtain

$$y^i . z^{-h} . dg = \Sigma_{-\infty}^{+\infty} \overset{(i)}{P_h} . dg.$$

Integrating between the limits $+\pi$ and $-\pi$, we have

$$\overset{(i)}{P_h} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} y^i . z^{-h} . dg \tag{17}$$

From

$$z = c^{g\sqrt{-1}} = \cos g + \sqrt{-1}\sin g,$$

we have

$$dz = (-\sin g + \sqrt{-1}\,.\cos g)\,dg\,;$$

also

$$z\sqrt{-1} = \sqrt{-1}\cos g - \sin g.$$

Therefore

$$dz = z\sqrt{-1}\,.\,dg,$$

and (17) becomes

$$\overset{(i)}{P}_h = \frac{1}{2\pi\sqrt{-1}}\int_{c^{-\pi\sqrt{-1}}}^{c^{+\pi\sqrt{-1}}} .\, y^i \,.\, z^{-h-1}\,.\, dz.$$

In like manner we find

$$\overset{(h)}{Q}_i = \frac{1}{2\pi\sqrt{-1}}\int_{c^{-\pi\sqrt{-1}}}^{c^{+\pi\sqrt{-1}}} z^h \,.\, y^{-i-1}\,.\, dy.$$

Integrating by parts we have

$$\overset{(h)}{Q}_i = \frac{1}{2\pi\sqrt{-1}}\,.\,\frac{h}{i}\int_{c^{-\pi\sqrt{-1}}}^{c^{+\pi\sqrt{-1}}} y^{-i}\,.\, z^{h-1}\,.\, dz. \tag{18}$$

Comparing this value of $\overset{(h)}{Q}_i$ with that of $\overset{(i)}{P}_h$, we obtain

$$i\,.\,\overset{(h)}{Q}_i = h\,.\,\overset{(-i)}{P}_{-h} = h\,.\,\overset{(i)}{P}_h,$$

or

$$\overset{(i)}{P}_h = \frac{i}{h}\,.\,\overset{(h)}{Q}_i = \frac{i}{h}\,.\,\overset{(h-i)}{J}_{h\lambda}\,. \tag{19}$$

Thus we have, between the mean and the eccentric anomaly, the relations

$$\left. \begin{aligned} z^h &= J_{h\lambda}^{(h-i)} \cdot y^i \\ y^i &= \tfrac{i}{h} \cdot J_{h\lambda}^{(h-i)} \cdot z^h \end{aligned} \right\} \tag{20}$$

In the application of these relations, since

$$y'^{-i'} = \Sigma P_{-h'}^{(-i')} \cdot z'^{-h'},$$

the expression for F is changed from

$$F = \tfrac{1}{2} \Sigma \Sigma \left\{ (i, i', c) - \sqrt{-1}\,(i, i', s) \right\} y^i \cdot y'^{-i'}$$

into

$$F = \tfrac{1}{2} \Sigma \Sigma \left\{ (i, i', c) - \sqrt{-1}\,(i, i', s) \right\} y^i \cdot \Sigma P_{-h'}^{-i'} \cdot z'^{-h'}.$$

The other value of F is

$$F = \tfrac{1}{2} \Sigma \Sigma \left\{ ((i, h', c)) - \sqrt{-1}\,((i, h', s)) \right. y^i \cdot z'^{-h'}.$$

A comparison of these two values gives

$$((i, h', c)) = \Sigma P_{-h'}^{(-i')} (i, i', c) = \Sigma \cdot \tfrac{i'}{h'} \cdot J_{h'\lambda'}^{(h'-i')} (i, i', c) \tag{21}$$

In transforming from the series indicated by (i, i', c) into that of $((i, h', c))$, it is evident that h' is constant in each individual case, and i' is the variable.

Thus we find, beginning with $i' = h'$,

$$((i, h', c)) = \tfrac{h'}{h'} \cdot J_{h'\lambda'}^{(h'-h')} (i, h', c) + \tfrac{h'-1}{h'} \cdot J_{h'\lambda'}^{(h'-(h'-1))} (i, h'-1, c) + \text{etc.}$$

$$+ \tfrac{h'+1}{h'} \cdot J_{h'\lambda'}^{(h'-(h'+1))} (i, h'+1, c) + \text{etc.}$$

To transform from $((i, h', c))$ into (i, i', c)
we have

$$(i, i', c) = \Sigma \overset{(-h')}{Q_{-i}} ((i, h', c)) = \Sigma \overset{(h'-i')}{J_{h'\lambda'}} ((i, h', c)).$$

Here, i' is the constant, and h' the variable; and for the different values of h', beginning with $h' = i'$,
we find

$$(i, i', c) = \overset{(0)}{J_{i'\lambda'}} ((i, i' c)) + \overset{((i'-1)-i'))}{J_{(i'-1)\lambda'}} ((i, i'-1, c)) + \text{etc.}$$
$$+ \overset{((i'+1)-i'))}{J_{(i'+1)\lambda'}} ((i, i'+1, c)) + \text{etc.}$$

The expression

$$\overset{(m)}{J_{h\lambda}} = \frac{h^m \lambda^m}{1..2..m} \left(1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4\lambda^4}{1.2.m+1.m+2} - \frac{h^6\lambda^6}{1.2.3.m+1.m+2.m+3} \pm \text{etc.}\right)$$

enables us to find the value of $\overset{(m)}{J_{h\lambda}}$ for all values of m.

A simpler method can be obtained in the following manner:

Putting $c^{h\frac{e}{2}(y-y^{-1})}$ in the form

$$c^{h\frac{e}{2}(y-y^{-1})} = \overset{(0)}{J_{h\frac{e}{2}}} + \overset{(1)}{J_{h\frac{e}{2}}} . y - \overset{(-1)}{J_{h\frac{e}{2}}} . y^{-1} + \overset{(2)}{J_{h\frac{e}{2}}} . y^2 + \overset{(-2)}{J_{h\frac{e}{2}}} . y^{-2} + \text{etc.}$$

we have, for the differential coefficient relative to y,

$$h\frac{e}{2}(1+y^{-2}) . c^{h\frac{e}{2}(y-y^{-1})} = \overset{(1)}{J_{h\frac{e}{2}}} + 2 . \overset{(2)}{J_{h\frac{e}{2}}} . y \pm \text{etc.} + \overset{(-1)}{J_{h\frac{e}{2}}} . y^{-2} - 2 \overset{(2)}{J_{h\frac{e}{2}}} y^{-3} \pm \text{etc.}$$

If we multiply the second member of the first equation by $h\frac{e}{2}(1 + y^{-2})$, we have an expression equal to the second member of the second expression, and by comparing the two we find

$$h\frac{e}{2} \left\{ \overset{(m+1)}{J_{h\frac{e}{2}}} + \overset{(m-1)}{J_{h\frac{e}{2}}} \right\} = m . \overset{(m)}{J_{h\frac{e}{2}}} \tag{22}$$

Let

$$\frac{J^{(m)}_{h\frac{e}{2}}}{J^{(m-1)}_{h\frac{e}{2}}} = p_m; \tag{23}$$

then

$$J^{(m)}_{h\frac{e}{2}} = J^{(m-1)}_{h\frac{e}{2}} \cdot p_m.$$

From this general expression we find

$$\begin{aligned} J^{(1)}_{h\frac{e}{2}} &= J^{(0)}_{h\frac{e}{2}} \cdot p_1 \\ J^{(2)}_{h\frac{e}{2}} &= J^{(1)}_{h\frac{e}{2}} \cdot p_2 = J^{(0)}_{h\frac{e}{2}} \cdot p_1 , p_2 \\ \text{etc.} &= \quad \text{etc.} \quad = \quad \text{etc.} \end{aligned} \tag{24}$$

From the values here given, since $\dfrac{J^{(m)}_{h\frac{e}{2}}}{J^{(m-1)}_{h\frac{e}{2}}}$ is put equal to p_m, we have, by increasing m by unity,

$$\frac{J^{(m+1)}_{h\frac{e}{2}}}{J^{(m-1)}_{h\frac{e}{2}}} = p_m \cdot p_{m+1}$$

Putting $\dfrac{m}{h\frac{e}{2}} = r_m$, equation (22)

takes the form

$$p_m \cdot p_{m+1} + 1 = r_m \cdot p_m.$$

From this we find

$$\begin{aligned} p_m &= \frac{1}{r_m - p_{m+1}} \\ &= \cfrac{1}{r_m - \cfrac{1}{r_{m+1} - \cfrac{1}{r_{m+2} - \text{etc.}}}} \end{aligned}$$

We also have

$$\frac{1}{p_m} = r_m - p_{m+1}, \tag{25}$$

a form more convenient in the applications.

The general expression for $\overset{(m)}{J}_{h\frac{e}{2}}$ is

$$\overset{(m)}{J}_{h\frac{e}{2}} = \overset{(0)}{J}_{h\frac{e}{2}} \cdot p_1 \cdot p_2 \cdot p_3 \ldots p_m, \tag{26}$$

where

$$\overset{(0)}{J}_{h\frac{e}{2}} = 1 - \frac{l^2}{1^2} + \frac{l^4}{1^2.2^2} - \frac{l^6}{1^2.2^2.3^2} \pm \text{etc.}, \tag{27}$$

if we put $l = h\lambda$.

From the expression

$$((i, h', c)) = \Sigma \overset{(-i')}{P}_{-h'} (i, i', c) = \Sigma \frac{i}{h'} \overset{(h'-i')}{J}_{h'\lambda'} (i, i', c)$$

it is evident that when $h' = 0$, or when both i' and h' are zero, this expression cannot be employed.

To find the values for these exceptional cases let us resume the equation

$$\overset{(i)}{P}_h = \frac{1}{2\pi\sqrt{-1}} \int_{c-\pi\sqrt{-1}}^{c+\pi\sqrt{-1}} y^i z^{-h-1} \, dz.$$

When $h = 0$ we have

$$\overset{(i)}{P}_0 = \frac{1}{2\pi\sqrt{-1}} \int_{c-\pi\sqrt{-1}}^{c+\pi\sqrt{-1}} y^i z^{-1} \, dz$$

The equation

$$z = y \, . \, c^{-\frac{e}{2}(y - y^{-1})}$$

gives

$$\frac{dz}{z} = \frac{dy}{y} - \frac{e}{2}(1 + y^{-2})\, dy. \tag{28}$$

Hence

$$\overset{(i)}{P_0} = \frac{1}{2\pi\sqrt{-1}} \int_{c^{-\pi\sqrt{-1}}}^{c^{+\pi\sqrt{-1}}} (y^{i-1} - \frac{e}{2}\, y^{i} - \frac{e}{2}\, y^{i-2})\, dy.$$

When p is a whole number

$$\int_{c^{-\pi\sqrt{-1}}}^{c^{+\pi\sqrt{-1}}} y^{p} . \, dy = 0,$$

except when $p = 1$, when this integral is $2\pi\sqrt{-1}$.
Hence it follows that

$$\overset{(1)}{P_0} = \overset{(-1)}{P_0} = -\tfrac{1}{2}\, e.$$

When $i = 0$, we have

$$\overset{(0)}{P_0} = 1.$$

Using the expression

$$\begin{aligned} ((i, h', c)) = \Sigma \, . \, \overset{(-i')}{P_{-h'}}(i, i', c) = \overset{(-i')}{P_{-h'}}(i, i', c) + \overset{(-i'-1)}{P_{-h'}}(i, i' + 1, c) \\ + \overset{(-i+1)}{P_{-h'}}(i, i' - 1, c), \end{aligned}$$

we have

$$((0, 0, c)) = (0, 0, c) - 2\lambda'\,(0, 1, c)$$

for the constant term, the double value of this term being employed.

For $h' = 0$, we have

$$((1,0,c)) = (1,0,c) - \lambda'(1,1,c) - \lambda'(1,-1,c)$$
$$((1,0,s)) = (1,0,s) - \lambda'(1,1,s) - \lambda'(1,-1,s)$$
$$((2,0,c)) = (2,0,c) - \lambda'(2,1,c) - \lambda'(2,-1,c)$$
$$((2,0,s)) = (2,0,s) - \lambda'(2,1,s) - \lambda'(2,-1,s)$$
$$\text{etc.} = \text{etc.}$$

In what precedes we have put

$g =$ the mean anomaly,
$\varepsilon =$ the eccentric anomaly,
$c =$ the Naperian base,
$z = c^{g\sqrt{-1}}$,
$y = c^{\varepsilon\sqrt{-1}}$,

and obtain

$$z^h = y^h . c^{-h\frac{e}{2}(y-y^{-1})},$$
$$y^i = z^i . c^{i\frac{e}{2}(y-y^{-1})};$$

where $c^{-h\frac{e}{2}(y-y^{-1})}$ is expressed in a series, the general term of which is

$$h^m \lambda^m \left(1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4\lambda^4}{1.2.m+1.m+2} - \frac{h^6\lambda^6}{1.2.3.m+1.m+2.m+3} \pm \text{etc.}\right) y^m.$$

Thus

$$z^h = y^h . h^m \lambda^m \left(1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4\lambda^4}{1.2.m+1.m+2} - \frac{h^6\lambda^6}{1.2.3.m+1.m+2.m+3} \pm \text{etc.}\right) y^m.$$

We have also put

$$c^{-h\frac{e}{2}(y-y^{-1})} = \Sigma_{-\infty}^{+\infty} \overset{(-m)}{J_{-h\lambda}} . y^{-m},$$

$$c^{h\frac{e}{2}(y-y^{-1})} = \Sigma_{-\infty}^{+\infty} \overset{(m)}{J_{h\lambda}} . y^m;$$

and since

$$\overset{(-m)}{J_{-h\lambda}} = \overset{(m)}{J_{h\lambda}},$$

have found

$$z^h = \overset{(m)}{J_{h\lambda}} \cdot y^{-m} \cdot y^h,$$
$$= \overset{(h-i)}{J_{h\lambda}} \cdot y^i,$$

if

$$m = h - i.$$

Again supposing

$$z^h = \Sigma_{-\infty}^{+\infty} \overset{(h)}{Q_i} \cdot y^i,$$
$$y^i = \Sigma_{-\infty}^{+\infty} \overset{(i)}{P_h} \cdot z^h,$$

we have found

$$\overset{(i)}{P_h} = \frac{i}{h} \cdot \overset{(h)}{Q_i} = \frac{i}{h} \cdot \overset{(h-i)}{J_{h\lambda}}.$$

Thus we have

$$z^h = \overset{(h-i)}{J_{h\lambda}} \cdot y^i,$$
$$= \overset{(h-i)}{J_{h\lambda}} \left[\cos i\varepsilon + \sin i\varepsilon \sqrt{-1}\right];$$
$$y^i = \frac{i}{h} \overset{(h-i)}{J_{h\lambda}} \cdot z^h$$
$$= \frac{i}{h} \cdot \overset{(h-i)}{J_{h\lambda}} \left[\cos hg + \sin hg \sqrt{-1}\right].$$

Equating real and imaginary terms, we have

$$\left.\begin{aligned} \cos i\varepsilon &= \frac{i}{h} \cdot \Sigma_{h=-\infty}^{h=\infty} \overset{(h-i)}{J_{h\lambda}} \cdot \cos hg, \\ \sin i\varepsilon &= \frac{i}{h} \cdot \Sigma_{h=-\infty}^{h=\infty} \overset{(h-i)}{J_{h\lambda}} \cdot \sin hg. \end{aligned}\right\} \qquad (29)$$

We notice that

$$\overset{(1)}{P_0} = \overset{(-1)}{P_0} = -\tfrac{1}{2}e,$$
$$\overset{(0)}{P_0} = 1.$$

For all other values of i

$$\overset{(i)}{P_0} = 0.$$

If a large number of the J functions are needed they are computed by means of equations (24) to (27), as shown in the example given in Chapter V.

If we wish to determine any of them independently we have from

$$\overset{(m)}{J_{h\lambda}} = \frac{h^m\lambda^m}{1.2...m}\left[1 - \frac{h^2\lambda^2}{1.m+1} + \frac{h^4.\lambda^4}{1.2.m+1.m+2} - \frac{h^6.\lambda^6}{1.2.3.m+1.m+2.m+3} \pm \text{etc.}\right],$$

$$\left.\begin{aligned}
\overset{(0)}{J_{h\frac{e}{2}}} &= \left[1 - \frac{h^2}{1}\cdot\frac{e^2}{4} + \frac{h^4}{4}\cdot\frac{e^4}{16} - \frac{h^6}{36}\cdot\frac{e^6}{64} \pm \text{etc.}\right] \\
\overset{(1)}{J_{h\frac{e}{2}}} &= \frac{h\frac{e}{2}}{1}\left[1 - \frac{h^2}{2}\cdot\frac{e^2}{4} + \frac{h^4}{12}\cdot\frac{e^4}{16} - \frac{h^6}{144}\cdot\frac{e^6}{64} \pm \text{etc.}\right] \\
\overset{(2)}{J_{h\frac{e}{2}}} &= \frac{(h.\frac{e}{2})^2}{1.2}\left[1 - \frac{h^2}{3}\cdot\frac{e^2}{4} + \frac{h^4}{24}\cdot\frac{e^4}{16} \mp \text{etc.}\right] \\
\overset{(3)}{J_{h\frac{e}{2}}} &= \frac{(h.\frac{e}{2})^3}{1.2.3}\left[1 - \frac{h^2}{4}\cdot\frac{e^2}{4} + \frac{h^4}{40}\cdot\frac{e^4}{16} \mp \text{etc.}\right] \\
\overset{(4)}{J_{h\frac{e}{2}}} &= \frac{(h.\frac{e}{2})^4}{1.2.3.4}\left[1 - \frac{h^2}{5}\cdot\frac{e^2}{4} \pm \text{etc.}\right]
\end{aligned}\right\} \quad (30)$$

In these expressions we have written for λ its value $\frac{1}{2}e$.

Since h has all values from $h = +\infty$ to $-\infty$ we find any value of $\overset{(m)}{J_{h\lambda}}$ by attributing proper values to h.

From equations (29) we find the values of the functions $\cos i\varepsilon$, $\sin i\varepsilon$, in terms of $\cos hg$, $\sin hg$, and the J functions just given; always noting that when $h = 0$, we have only for $i = \pm 1$, $-\frac{1}{2}e$ as the value of the function.

We can employ equation (22) when only a few functions are needed, or as a check.

It may be of value to have y^i in terms of z^h and the J functions. From the second of equations (20) we have

$$y^{+1} = -\lambda + \overset{(0)}{J_{\lambda}} . z + \tfrac{1}{2}\overset{(1)}{J_{2\lambda}} . z^2 + \tfrac{1}{3}\overset{(2)}{J_{3\lambda}} . z^3 + \text{etc.}$$
$$- \overset{(2)}{J_{\lambda}} . z^{-1} - \tfrac{1}{2}\overset{(3)}{J_{2\lambda}} . z^{-2} - \tfrac{1}{3}\overset{(4)}{J_{3\lambda}} . z^{-3} - \text{etc.}$$

$$y^{-1} = -\lambda + \overset{(0)}{J_{\lambda}} . z^{-1} + \tfrac{1}{2}\overset{(1)}{J_{2\lambda}} . z^{-2} + \tfrac{1}{3}\overset{(2)}{J_{3\lambda}} . z^{-3} + \text{etc.}$$
$$- \overset{(2)}{J_{\lambda}} . z - \tfrac{1}{2}\overset{(3)}{J_{2\lambda}} . z^2 - \tfrac{1}{3}\overset{(4)}{J_{3\lambda}} . z^3 - \text{etc.}$$

$$y^{+2} = -\tfrac{2}{1}\overset{(1)}{J_{\lambda}} . z + \tfrac{2}{2}\overset{(0)}{J_{2\lambda}} . z^2 + \tfrac{2}{3}\overset{(1)}{J_{3\lambda}} . z^3 + \text{etc.}$$
$$- \tfrac{2}{1}\overset{(3)}{J_{\lambda}} . z^{-1} - \tfrac{2}{2}\overset{(4)}{J_{2\lambda}} . z^{-2} - \tfrac{2}{3}\overset{(5)}{J_{3\lambda}} . z^{-3} - \text{etc.}$$

$$y^{-2} = -\tfrac{2}{1}\overset{(1)}{J_{\lambda}} . z^{-1} + \tfrac{2}{2}\overset{(0)}{J_{2\lambda}} . z^{-2} + \tfrac{2}{3}\overset{(1)}{J_{3\lambda}} . z^{-3} + \text{etc.}$$
$$- \tfrac{2}{1}\overset{(3)}{J_{\lambda}} . z - \tfrac{2}{2}\overset{(4)}{J_{2\lambda}} . z^2 - \tfrac{2}{3}\overset{(5)}{J_{3\lambda}} . z^{-3} - \text{etc.}$$

Then from

$$y^i + y^{-i} = 2 \cos i\varepsilon$$
$$y^i - y^{-i} = 2 \sqrt{-1} . \sin i\varepsilon$$

we find the values of $\cos \varepsilon$, $\sin \varepsilon$, $\cos 2\varepsilon$, $\sin 2\varepsilon$, etc.

In case of the sine, as for example when $i = 1$, we have

$$y - y^{-1} = 2 \sqrt{-1} \sin \varepsilon \text{; but in } z - z^{-1} = 2 \sqrt{-1} \sin g,$$

we have the same factor, $2 \sqrt{-1}$, in the second member of the equation.

From

$$r = a(1 - e \cos \varepsilon)$$

we find

$$\left(\frac{r}{a}\right)^2 = 1 - 2e \cos \varepsilon + e^2 \cos {}^2\varepsilon$$

$$\left(\frac{a}{r}\right)^2 = 1 + 2e \cos \varepsilon + 3e^2 \cos {}^2\varepsilon + 4e^3 \cos {}^3\varepsilon + \text{etc.}$$

For $\left(\frac{r}{a}\right)^2$ we have

$$\left(\frac{r}{a}\right)^2 = 1 + \tfrac{1}{2}e^2 - 2e \cos \varepsilon + \tfrac{1}{2}e^2 \cos 2\varepsilon$$

But

$$\frac{d}{dg}\left(\frac{r^2}{a^2}\right) = 2e \sin \varepsilon \,(1 - e \cos \varepsilon\,) \frac{d\varepsilon}{dg} = 2e \sin \varepsilon,$$

and

$$\sin \varepsilon = \left[\overset{(0)}{J_\lambda} + \overset{(2)}{J_\lambda}\right] \sin g + \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda}} + \overset{(3)}{J_{2\lambda}}\right] \sin 2g + \tfrac{1}{3}\left[\overset{(2)}{J_{3\lambda}} + \overset{(4)}{J_{3\lambda}}\right] \sin 3g + \text{etc.}$$

Multiplying by $2e\,.\,dg$ we have for the integral of $\frac{d}{dg}\left(\frac{r^2}{a^2}\right)$

$$\frac{r^2}{a^2} = c - \frac{2e}{1}\left[\overset{(0)}{J_\lambda} + \overset{(2)}{J_\lambda}\right] \cos g - \frac{2e}{4}\left[\overset{(1)}{J_{2\lambda}} + \overset{(3)}{J_{2\lambda}}\right] \cos 2g - \frac{2e}{9}\left[\overset{(2)}{J_{3\lambda}} + \overset{(4)}{J_{3\lambda}}\right] \cos 3g - \text{etc.}$$

where $c = 1 + \frac{3}{2}e^2$.

By means of (22) this becomes

$$\left(\frac{r}{a}\right)^2 = 1 + \tfrac{3}{2}e^2 - \tfrac{4}{1}\overset{(1)}{J_\lambda} \cos g - \tfrac{4}{4}\,\overset{(2)}{J_{2\lambda}} \cos 2g - \tfrac{4}{9}\overset{(3)}{J_{3\lambda}} \cos 3g - \text{etc.}$$

In case of $\left(\frac{r}{a}\right)^{-2}$, we have

$3e^2\,.\cos^2\varepsilon = \frac{3}{2}e^2\,(1 + \cos 2\varepsilon)$, $4e^3 \cos^3\varepsilon = e^3\,(3 \cos \varepsilon + \cos 3\varepsilon)$,

$5e^4\,.\cos^4\varepsilon = \frac{5}{8}e^4\,(3 + 4\cos 2\varepsilon + \cos 4\varepsilon)$, $6e^5\,.\cos^5\varepsilon = \frac{6}{16}e^5\,(10 \cos \varepsilon + 5 \cos 3\varepsilon + \cos 5\varepsilon)$,

$7e^6 \cos^6\varepsilon = \frac{7}{32}e^6\,(10 + 15 \cos 2\varepsilon + 6 \cos 4\varepsilon + \text{etc.})$

and hence

$$\begin{aligned}\left(\frac{r}{a}\right)^{-2} = 1 &+ \tfrac{3}{2}e^2 + \tfrac{15}{8}e^4 + \tfrac{70}{32}e^6 + \text{etc.}\\ &+ [2e + 3e^3 + \tfrac{60}{16}e^5 + \text{etc.}] \cos \varepsilon\\ &+ [\tfrac{3}{2}e^2 + \tfrac{20}{8}e^4 + \tfrac{105}{32}e^6 + \text{etc.}] \cos 2\varepsilon\\ &+ [e^3 + \tfrac{30}{16}e^5 + \text{etc.}] \cos 3\varepsilon\\ &+ [\tfrac{5}{8}e^4 + \tfrac{42}{32}e^6 + \text{etc.}] \cos 4\varepsilon\end{aligned}$$

Attributing to i proper values in equation (29) we find the expressions for $\cos\varepsilon$, $\cos 2\varepsilon$, $\cos 3\varepsilon$, etc. We then multiply these expressions by their appropriate factors and thus have the value of $\left(\frac{r}{a}\right)^{-2}$. Let

$$\left(\frac{r}{a}\right)^2 = \Sigma_{-\infty}^{+\infty} \overset{(2)}{R_i} \cos ig, \qquad \left(\frac{r}{a}\right)^{-2} = \Sigma_{-\infty}^{+\infty} \overset{(-2)}{R_i} \cos ig.$$

The following are the values of $\overset{(2)}{R_i}$ and $\overset{(-2)}{R_i}$ to terms of the seventh order of e.

$$\overset{(2)}{R_0} = 1 + \tfrac{3}{2}e^2$$

$$\overset{(2)}{R_1} = -2e + \tfrac{1}{4}e^3 - \tfrac{1}{96}e^5 + \tfrac{1}{4608}e^7$$

$$\overset{(2)}{R_2} = -\tfrac{1}{2}e^2 + \tfrac{1}{6}e^4 - \tfrac{1}{48}e^6$$

$$\overset{(2)}{R_3} = -\tfrac{1}{4}e^3 + \tfrac{9}{64}e^5 - \tfrac{81}{2560}e^7$$

$$\overset{(2)}{R_4} = -\tfrac{1}{6}e^4 + \tfrac{2}{15}e^6$$

$$\overset{(2)}{R_5} = -\tfrac{25}{192}e^5 + \tfrac{625}{4608}e^7$$

$$\overset{(2)}{R_6} = -\tfrac{9}{80}e^6$$

$$\overset{(2)}{R_7} = -\tfrac{2401}{23040}e^7.$$

$$\overset{(-2)}{R_0} = \frac{1}{\sqrt{1-e^2}} = 1 + e^2 + \tfrac{3}{4}e^4 + \tfrac{15}{8}e^6 + \text{etc.}$$

$$\overset{(-2)}{R_1} = 2e + \tfrac{3}{4}e^3 + \tfrac{65}{96}e^5 + \tfrac{2675}{4608}e^7$$

$$\overset{(-2)}{R_2} = \tfrac{5}{2}e^2 + \tfrac{1}{3}e^4 + \tfrac{21}{32}e^6$$

$$\overset{(-2)}{R_3} = \tfrac{13}{4}e^3 - \tfrac{25}{64}e^5 + \tfrac{393}{512}e^7$$

$$\overset{(-2)}{R_4} = \tfrac{103}{24}e^4 - \tfrac{387}{240}e^6$$

$$\overset{(-2)}{R_5} = \tfrac{1097}{192}e^5 - \tfrac{16621}{4608}e^7$$

$$\overset{(-2)}{R_6} = \tfrac{1223}{160}e^6$$

$$\overset{(-2)}{R_7} = \tfrac{47273}{4608}e^7.$$

See HANSEN'S *Fundamenta nova*, pp. 172, 173.

We add also the differential coefficients of $\overset{(2)}{R_i}$, $\overset{(-2)}{R_i}$, relative to e.

$$\frac{d\overset{(2)}{R_0}}{de} = 3e$$

$$\frac{d\overset{(2)}{R_1}}{de} = -2 + \tfrac{3}{4}e^2 - \tfrac{5}{96}e^4 + \tfrac{7}{4608}e^6 \mp \text{etc.}$$

$$\frac{d\overset{(2)}{R_2}}{de} = -e + \tfrac{2}{3}e^3 - \tfrac{1}{8}e^5 \pm \text{etc.}$$

$$\frac{d\overset{(2)}{R_3}}{de} = -\tfrac{3}{4}e^2 + \tfrac{45}{64}e^4 - \tfrac{567}{2560}e^6 \pm \text{etc.}$$

$$\frac{d\overset{(2)}{R_4}}{de} = -\tfrac{2}{3}e^3 + \tfrac{4}{5}e^5 \mp \text{etc.}$$

$$\frac{d\overset{(2)}{R_5}}{de} = -\tfrac{125}{192}e^4 + \tfrac{4375}{4608}e^6 \mp \text{etc.}$$

$$\frac{d\overset{(2)}{R_6}}{de} = -\tfrac{27}{40}e^5 \pm \text{etc.}$$

$$\frac{d\overset{(2)}{R_7}}{de} = -\tfrac{16807}{23040}e^6 \pm \text{etc.}$$

$$\text{etc.} = \text{etc.}$$

$$\frac{d\overset{(-2)}{R_0}}{de} = e + 3e^3 + \tfrac{45}{4}e^5 + \tfrac{105}{2}e^7$$

$$\frac{d\overset{(-2)}{R_1}}{de} = 2 + \tfrac{9}{4}e^2 + \tfrac{325}{96}e^4 + \tfrac{18725}{4608}e^6$$

$$\frac{d\overset{(-2)}{R_2}}{de} = 5e + \tfrac{4}{3}e^3 + \tfrac{63}{16}e^5$$

$$\frac{d\overset{(-2)}{R_3}}{de} = \tfrac{39}{4}e^2 - \tfrac{125}{64}e^4 + \tfrac{2751}{512}e^6$$

$$\frac{d\overset{(-2)}{R_4}}{de} = \tfrac{103}{6}e^3 - \tfrac{387}{40}e^5$$

$$\frac{d\overset{(-2)}{R_5}}{de} = \tfrac{5485}{192}e^4 - \tfrac{116347}{4608}e^6$$

$$\frac{d\overset{(-2)}{R_6}}{de} = \tfrac{3669}{80}e^5$$

$$\frac{d\overset{(-2)}{R_7}}{de} = \tfrac{330911}{4608}e^6$$

The value of $\frac{r^2}{a^2}$ found by integrating $d\left(\frac{r^2}{a^2}\right) = 2e \,.\, \sin \varepsilon \,.\, dg$, is

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - \tfrac{4}{1}\overset{(1)}{J_\lambda} \cos g - \tfrac{4}{4}\overset{(2)}{J_{2\lambda}} \cos 2g - \tfrac{4}{9}\overset{(3)}{J_{3\lambda}} \cos 3g - \text{etc.}$$

In terms of the $\overset{(2)}{R_i}$ functions,

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - \overset{(2)}{R_1} \cos g - \overset{(2)}{R_2} \cos 2g - \overset{(2)}{R_3} \cos 3g - \text{etc.}$$

Again, since

$$\frac{df}{dg} = \frac{a^2}{r^2} \sqrt{1 - e^2},$$

we have

$$\frac{a^2}{r^2} = \overset{(-2)}{R_i} \cos ig = \frac{1}{\sqrt{1 - e^2}} \cdot \frac{df}{dg}.$$

Let

$$f = g + \Sigma_{+1}^{+\infty} C_i \sin ig\,;$$

then

$$\frac{df}{dg} = 1 + \Sigma_{+1}^{+\infty} i C_i \cos ig,$$

and hence

$$\overset{(-2)}{R_i} = \frac{i \,.\, C_i}{\sqrt{1 - e^2}}.$$

The coefficients represented by C_i designate the coefficients of the equation of the centre.

Using the values of the C_i coefficients given by LE VERRIER in the *Annales de l'Observatoire Impérial de Paris*, Tome Premier, p. 203, we have

$$
\begin{aligned}
f-g = & \left[4\left(\tfrac{e}{2}\right)-2\left(\tfrac{e}{2}\right)^3+\tfrac{5}{3}\left(\tfrac{e}{2}\right)^5+\tfrac{107}{36}\left(\tfrac{e}{2}\right)^7+\tfrac{6217}{720}\left(\tfrac{e}{2}\right)^9\right]\sin g \\
& +\left[5\left(\tfrac{e}{2}\right)^2-\tfrac{22}{3}\left(\tfrac{e}{2}\right)^4+\tfrac{17}{3}\left(\tfrac{e}{2}\right)^6+\tfrac{86}{45}\left(\tfrac{e}{2}\right)^8+\text{etc.}\right]\sin 2g \\
& +\left[\tfrac{26}{3}\left(\tfrac{e}{2}\right)^3-\tfrac{43}{2}\left(\tfrac{e}{2}\right)^5+\tfrac{95}{4}\left(\tfrac{e}{2}\right)^7-\tfrac{973}{120}\left(\tfrac{e}{2}\right)^9+\text{etc.}\right]\sin 3g \\
& +\left[\tfrac{103}{6}\left(\tfrac{e}{2}\right)^4-\tfrac{902}{15}\left(\tfrac{e}{2}\right)^6+\tfrac{4123}{45}\left(\tfrac{e}{2}\right)^8-\text{etc.}\right]\sin 4g \\
& +\left[\tfrac{1097}{30}\left(\tfrac{e}{2}\right)^5-\tfrac{5957}{36}\left(\tfrac{e}{2}\right)^7+\tfrac{164921}{504}\left(\tfrac{e}{2}\right)^9\right]\sin 5g \\
& +\left[\tfrac{1223}{15}\left(\tfrac{e}{2}\right)^6-\tfrac{15826}{35}\left(\tfrac{e}{2}\right)^8+\text{etc.}\right]\sin 6g \\
& +\left[\tfrac{47273}{252}\left(\tfrac{e}{2}\right)^7-\tfrac{1773271}{1440}\left(\tfrac{e}{2}\right)^9\right]\sin 7g \\
& +\left[\tfrac{556403}{1260}\left(\tfrac{e}{2}\right)^8\right]\sin 8g \\
& +\left[\tfrac{10661993}{10080}\left(\tfrac{e}{2}\right)^9\right]\sin 9g
\end{aligned}
$$

Converting the coefficients into seconds of arc, and writing the logarithms of the numbers, we have for the equation of the centre,

$$
\begin{aligned}
& f-g= \\
& +\left[5.9164851\left(\tfrac{e}{2}\right)-5.6154551\left(\tfrac{e}{2}\right)^3+5.5362739\left(\tfrac{e}{2}\right)^5+5.787506\left(\tfrac{e}{2}\right)^7+6.25067\left(\tfrac{e}{2}\right)^9\right]\sin g \\
& +\left[6.0133951\left(\tfrac{e}{2}\right)^2-6.1797266\left(\tfrac{e}{2}\right)^4+6.067753\left(\tfrac{e}{2}\right)^6+5.59571\left(\tfrac{e}{2}\right)^8\right]\sin 2g \\
& +\left[6.2522772\left(\tfrac{e}{2}\right)^3-6.6468636\left(\tfrac{e}{2}\right)^5+6.690089\left(\tfrac{e}{2}\right)^7-6.22336\left(\tfrac{e}{2}\right)^9\right]\sin 3g \\
& +\left[6\ 5491111\left(\tfrac{e}{2}\right)^4-7.093540\left(\tfrac{e}{2}\right)^6+7.27643\left(\tfrac{e}{2}\right)^8\right]\sin 4g \\
& +\left[6.8775105\left(\tfrac{e}{2}\right)^5-7.533150\left(\tfrac{e}{2}\right)^7+7.82927\left(\tfrac{e}{2}\right)^9\right]\sin 5g \\
& +\left[7.225760\left(\tfrac{e}{2}\right)^6-7.96973\left(\tfrac{e}{2}\right)^8\right]\sin 6g \\
& +\left[7.587638\left(\tfrac{e}{2}\right)^7-8.40484\left(\tfrac{e}{2}\right)^9\right]\sin 7g \\
& +\left[7.95944\left(\tfrac{e}{2}\right)^8\right]\sin 8g \\
& +\left[8.33880\left(\tfrac{e}{2}\right)^9\right]\sin 9g
\end{aligned}
$$

CHAPTER III.

Development of the Perturbing Function and the Disturbing Forces.

By means of the formulæ given in the preceding chapter, the functions $\mu . \left(\frac{a}{\Delta}\right)$, $\mu . a^2\left(\frac{a}{\Delta}\right)^3$, etc., can be put in the desired form. The next step is to determine the complete expression for the perturbing function, and also the expressions for the disturbing forces.

If k^2 is taken as the measure of the mass of the Sun, and m the relation between the mass of the Sun and that of a planet, the mass of the planet is represented by mk^2.

If x, y, z, be the rectangular coördinate of a body, those of the disturbing body being expressed by the same letters with accents, the perturbing function is given in the form

$$\Omega = \frac{m'}{1+m}\left[\frac{1}{\Delta} - \frac{xx'+yy'+zz'}{r'^3}\right]$$

Now

$$\Delta^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$
$$= r^2 + r'^2 - 2rr' . H;$$

hence

$$a\,\Omega = \frac{m'}{1+m}\left[\frac{a}{\Delta} - \frac{ar}{r'^2} . H\right]$$

If $a\,\Omega$ is regarded as expressed in seconds of arc, and if we put

$$s = 206264''.8, \quad \mu = \frac{m'}{1+m} . s, \quad \alpha = \frac{a'}{a}, \quad (H) = \frac{\mu}{\alpha^2} . \left(\frac{a'}{r'}\right)^2 . \left(\frac{r}{a}\right) . H,$$

we have

$$a\,\Omega = \mu . \left(\frac{a}{\Delta}\right) - (H).$$

Finding the expression for (H) first by the method of HANSEN, we let

$$h = \frac{\mu}{a^2} . k . \cos(\Pi - K), \qquad h' = \frac{\mu}{a^2} . \cos\phi . \cos\phi' . k_1 . \cos(\Pi - K_1)$$

$$l = \frac{\mu}{a^2} . \cos\phi . k . \sin(\Pi - K), \; l' = \frac{\mu}{a^2} . \cos\phi' . k_1 . \sin(\Pi - K_1),$$

and have, if we make use of the eccentric anomaly,

$$(H) = h . \cos\varepsilon \left(\frac{a'}{r'}\right)^2 . \cos f' - eh\left(\frac{a'}{r'}\right)^2 . \cos f' - l . \sin\varepsilon . \left(\frac{a'}{r'}\right)^2 . \cos f'$$

$$+ l' . \cos\varepsilon \left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'} - el'\left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'} + h' . \sin\varepsilon \left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'}$$

Putting

$$\left(\frac{a'}{r'}\right)^2 \cos f' = \gamma'_1 . \cos g' + \gamma'_2 . \cos 2g' + \gamma'_3 . \cos 3g' + \text{etc.}$$

$$\left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'} = \delta'_1 . \sin g' + \delta'_2 . \sin 2g' + \delta'_3 . \sin 3g' + \text{etc.}$$

we find

$$\left.\begin{aligned}
(H) = {} & \tfrac{1}{2}(h\gamma'_1 - h'\delta'_1)\cos(-g' - \varepsilon) + \tfrac{1}{2}(l\gamma'_1 - l'\delta'_1)\sin(-g' - \varepsilon) \\
& - eh\gamma'_1 \cos(-g') + el'\delta'_1 \sin(-g') \\
& + \tfrac{1}{2}(h\gamma'_1 + h'\delta'_1)\cos(g' - \varepsilon) + \tfrac{1}{2}(l\gamma'_1 + l'\delta'_1)\sin(g' - \varepsilon) \\
& + 2(h\gamma'_2 - h'\delta'_2)\cos(-2g' - \varepsilon) + 2(l\gamma'_2 - l'\delta'_2)\sin(-2g' - \varepsilon) \\
& - 4.eh\gamma'_2 \cos(-2g') + 4.el'\delta'_2 \sin(-2g') \\
& + 2(h\gamma'_2 + h'\delta'_2)\cos(2g' - \varepsilon) + 2(l\gamma'_2 + l'\delta'_2)\sin(2g' - \varepsilon) \\
& + \text{etc.} \qquad + \text{etc.},
\end{aligned}\right\} \quad (1)$$

where

$$\delta'_1 = \overset{(0)}{J_{\lambda'}} + \overset{(2)}{J_{\lambda'}}, \qquad \gamma'_1 = \overset{(0)}{J_{\lambda'}} - \overset{(2)}{J_{\lambda'}}$$

$$\delta'_2 = \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}} + \overset{(3)}{J_{2\lambda'}}\right], \qquad \gamma'_2 = \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}} - \overset{(3)}{J_{2\lambda'}}\right]$$

etc. etc.

When the numerical value of (H) has been found from this equation we transform it into another in which both the angles involved are mean anomalies. For this purpose we compute the values of the J functions depending on the eccentricity, e, of the disturbed body just as has been done for the disturbing body. The values of the J functions can be checked by means of the values of $\overset{(0)}{J}_{h\lambda}$, $\overset{(1)}{J}_{h\lambda}$, given in ENGELMAN'S edition of the *Abhandlungen von Friedrich Wilhelm Bessel*, Erster Band, seite 103–109, or by equations $(30)_2$.

Thus by means of the equation

$$\overset{(m+1)}{J}_{h\lambda} + \overset{(m-1)}{J}_{h\lambda} = \frac{m}{h.\lambda} \,.\, \overset{(m)}{J}_{\lambda}$$

we are enabled to find $\overset{(m)}{J}_{h\lambda}$ if $\overset{(0)}{J}_{h\lambda}$, $\overset{(1)}{J}_{h\lambda}$ are known.

It must be noted that the argument of BESSEL'S table is $2\,.\,h\frac{e}{2}$, or $2\,.\,h\lambda$, or he. Thus if it is sought to find the value of $\overset{(1)}{J}_{2\lambda}$, we enter the table with $2\,.\,2\lambda$ or $2e$ as the argument.

When we need the functions for h from $h = -1$ to $h = 4$, we must find the values of $\frac{1}{4}\overset{(3)}{J}_{4\frac{e}{2}}$, $\frac{1}{3}\overset{(2)}{J}_{3\frac{e}{2}}$, $\frac{1}{2}\overset{(1)}{J}_{2\frac{e}{2}}$, $\frac{1}{1}\overset{(0)}{J}_{\frac{e}{2}}$, $-\frac{e}{2}$, and $-\frac{1}{1}\overset{(-2)}{J}_{-\frac{e}{2}}$.

The values of $\frac{1}{2}\,.\,\overset{(1)}{J}_{2\,.\,\frac{e}{2}}$ and $\overset{(0)}{J}_{\frac{e}{2}}$ we take from the table. To find $\overset{(3)}{J}_{4\frac{e}{2}}$ we have

$$\begin{aligned}\overset{(3)}{J}_{4\frac{e}{2}} &= -\overset{(1)}{J}_{4\frac{e}{2}} + \frac{2}{4\,.\,\frac{e}{2}}\,.\,\overset{(2)}{J}_{4\frac{e}{2}} \\ &= -\overset{(1)}{J}_{4\frac{e}{2}} + \frac{2}{4\,.\,\frac{e}{2}}\left[-\overset{(0)}{J}_{4\frac{e}{2}} + \frac{1}{4\frac{e}{2}}\,.\,\overset{(1)}{J}_{4\frac{e}{2}}\right]\end{aligned}$$

For $\overset{(2)}{J}_{3\frac{e}{2}}$ we have

$$\overset{(2)}{J}_{3\frac{e}{2}} = -\overset{(0)}{J}_{3\frac{e}{2}} + \frac{1}{3\,.\,\frac{e}{2}}\,.\,\overset{(1)}{J}_{3\frac{e}{2}}.$$

And for $\overset{(2)}{J}_{\frac{e}{2}}$ we have

$$\overset{(2)}{J}_{\frac{e}{2}} = -\overset{(0)}{J}_{\frac{e}{2}} + \frac{1}{\frac{e}{2}}\overset{(1)}{J}_{\frac{e}{2}}\,.$$

The expression for (H) can be put in a form in which both the angles are mean anomalies. Thus, resuming the expression for (H),

$$(H) = h \,.\cos\varepsilon \left(\frac{a'}{r'}\right)^2 \cos f' - eh \left(\frac{a'}{r'}\right)^2 \cos f' - l\,.\sin\varepsilon \left(\frac{a'}{r'}\right)^2 .\cos f'$$

$$+ l'\,.\cos\varepsilon \left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'} - el'\left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'} + h'\,.\sin\varepsilon\,.\left(\frac{a'}{r'}\right)^2 . \frac{\sin f'}{\cos\varphi'},$$

in which

$$h = \frac{\mu}{a^2}.k.\cos(\Pi - K)$$

$$h' = \frac{\mu}{a^2}.\cos\phi\,.\cos\phi'\,.k_1.\cos(\Pi - K_1) = \tfrac{1}{2}\mu\,.\frac{v\cos V}{a^3}$$

$$l = \frac{\mu}{a^2}.\cos\phi\,. \qquad k\,.\sin(\Pi - K) = \tfrac{1}{2}\mu\,.\frac{v\sin V}{a^3}$$

$$l' = \frac{\mu}{a^2}.\cos\phi'\,. \qquad k_1.\sin(\Pi - K_1) = \tfrac{1}{2}\mu\,.\frac{p\cos P}{a^3},$$

we find the expressions for $\left(\frac{a'}{r'}\right)^2\cos f'$, $\left(\frac{a'}{r'}\right)^2 \frac{\sin f'}{\cos\varphi'}$, as follows. We put as before

$$\left(\frac{a'}{r'}\right)^2 \cos f' = \gamma'_1\cos g' + \gamma'_2\cos 2g' + \gamma'_3\cos 3g' + \text{etc.}$$

$$\left(\frac{a'}{r'}\right)^2 \frac{\sin f'}{\cos\varphi'} = \delta'_1\sin g' + \delta'_2\sin 2g' + \delta'_3\sin 3g' + \text{etc.}$$

If we differentiate $\frac{r'}{a'}\cos f$ relative to g' we have

$$\frac{d\left(\frac{r'}{a'}.\cos f'\right)}{dg'} = \frac{\cos f'}{a'}.\frac{dr'}{dg'} - \frac{r'}{a'}.\sin f'.\frac{df'}{dg'} = -\frac{\sin f'}{\cos\varphi'},$$

since $$\frac{dr'}{dg'} = \frac{a'e'\sin f'}{\cos\varphi'}, \qquad \frac{df'}{dg'} = \frac{a'^2}{r'^2}.\cos\phi;$$

and hence

$$\frac{d^2\left(\frac{r'}{a'}\cos f'\right)}{dg'^2} = -\frac{a'^2}{r'^2}.\cos f'.$$

Similarly, in the case of $\frac{r'}{a'}\frac{\sin f'}{\cos\varphi'}$, we have

$$\frac{d^2}{dg'^2}\left(\frac{r'}{a'}\frac{\sin f'}{\cos\varphi'}\right)=-\frac{a'^2}{r'^2}\cdot\frac{\sin f'}{\cos\varphi'}.$$

But $\frac{r'}{a'}\cos f'=\cos\varepsilon'-e'$, and $\frac{r'}{a'}\frac{\sin f'}{\cos\varphi'}=\sin\varepsilon'$.

Hence

$$-\frac{d^2(\frac{r'}{a'}\cos f')}{dg'^2}=\frac{a'^2}{r'^2}\cos f'=-\frac{d^2.\cos\varepsilon'}{dg'^2},$$

$$-\frac{d^2\left(\frac{r'}{a'}\frac{\sin f'}{\cos\varphi'}\right)}{dg'^2}=\frac{a'^2}{r'^2}\cdot\frac{\sin f'}{\cos\varphi'}=-\frac{d^2.\sin\varepsilon'}{dg'^2}.$$

Now

$$\cos\varepsilon'=-\lambda'+\left[\overset{(0)}{J_{\lambda'}}-\overset{(2)}{J_{\lambda'}}\right]\cos g'+\tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}}-\overset{(3)}{J_{2\lambda'}}\right]\cos 2g'+\text{etc.}$$

$$\sin\varepsilon'=\left[\overset{(0)}{J_{\lambda'}}+\overset{(2)}{J_{\lambda'}}\right]\sin g'+\tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}}+\overset{(3)}{J_{2\lambda'}}\right]\sin 2g'+\text{etc.}$$

From the values of $\cos\varepsilon'$ and $\sin\varepsilon'$ we have

$$\frac{a'^2}{r'^2}\cos f'=\left[\overset{(0)}{J_{\lambda'}}-\overset{(2)}{J_{\lambda'}}\right]\cos g'+2\left[\overset{(1)}{J_{2\lambda'}}-\overset{(3)}{J_{2\lambda'}}\right]\cos 2g'+3\left[\overset{(2)}{J_{3\lambda'}}-\overset{(4)}{J_{3\lambda'}}\right]\cos 3g'+\text{etc.}$$

$$\frac{a'^2}{r'^2_{\,,}}\frac{\sin f'}{\cos\varphi}=\left[\overset{(0)}{J_{\lambda'}}+\overset{(2)}{J_{\lambda'}}\right]\sin g'+2\left[\overset{(1)}{J_{2\lambda'}}+\overset{(3)}{J_{2\lambda'}}\right]\sin 2g'+3\left[\overset{(2)}{J_{3\lambda'}}+\overset{(4)}{J_{3\lambda'}}\right]\sin 3g'+\text{etc.}$$

We now assume

$$\gamma_i=\frac{1}{i}\left[\overset{(i-1)}{J_{i\lambda}}-\overset{(i+1)}{J_{i\lambda}}\right],\qquad \delta_i=\frac{1}{i}\left[\overset{(i-1)}{J_{i\lambda}}+\overset{(i+1)}{J_{i\lambda}}\right]$$

$$\gamma'_{i'}=\frac{1}{i'}\left[\overset{(i'-1)}{J_{i'\lambda'}}-\overset{(i'+1)}{J_{i'\lambda'}}\right],\qquad \delta'_{i'}=\frac{1}{i'}\left[\overset{(i'-1)}{J_{i'\lambda'}}+\overset{(i'+1)}{J_{i'\lambda'}}\right].$$

Comparing these expressions for $\gamma'_{i'}$, $\delta'_{i'}$, with those found in the expression for $\frac{a'^2}{r'^2}\cdot\frac{\sin f'}{\cos\varphi'}$ given above, we see that the relation between them is i'^2.

The expressions for $\cos\varepsilon$, $\sin\varepsilon$, are the same as those of $\cos\varepsilon'$, $\sin\varepsilon'$, if we omit the accents.

Hence if we perform the operations indicated in the expression for (H), we have

$$(H)=\frac{\mu}{a^2}\cdot\left(\frac{a'}{r'}\right)^2\cdot\frac{r}{a}\cdot H$$

$$=\tfrac{1}{2}i'^2\left[h\gamma_i\gamma'_{i'}\pm h'\delta_i\,\delta'_{i'}\right]\cos(\pm ig-i'g')-\tfrac{1}{2}i'^2\left[l\delta_i\gamma'_{i'}\pm l'\gamma_i\delta'_{i'}\right]\sin(\pm ig-i'g')\qquad(2)$$

i and i' having all positive values.

Attributing to i and i' particular values, we find, noting that $\delta_0=0$, and $\delta'_0=0'$,

$$\begin{aligned}
(H)=\;&\tfrac{1}{2}\left[h.\gamma_1\gamma'_1+h'\delta_1\delta'_1\ \right]\cos(\ g-\ g')-\tfrac{1}{2}\left[l\delta_1\gamma'_1+l'\gamma_1\delta'_1\right]\sin(\ g-\ g')\\
&+\tfrac{1}{2}\left[h.\gamma_1\gamma'_1-h'\delta_1\delta'_1\ \right]\cos(-g-\ g')-\tfrac{1}{2}\left[l\delta_1\gamma'_1-l'\gamma_1\delta'_1\right]\sin(-g-\ g')\\
&+\tfrac{1}{2}h.\gamma_0.\gamma'_1\qquad\cos(\ -\ g')-\tfrac{1}{2}l'\gamma_0\delta'_1\qquad\sin(\ -\ g')\\
&+2\left[h.\gamma_1\gamma'_2+h'.\delta_1\delta'_2\right]\cos(\ g-2g')-2\left[l.\delta_1\gamma'_2+l'\gamma_1\delta'_2\right]\sin(\ g-2g')\\
&+2\left[h.\gamma_1\gamma'_2-h'.\delta_1\delta'_2\right]\cos(-g-2g')-2\left[l.\delta_1\gamma'_2-l'\gamma_1\delta'_2\right]\sin(-g-2g')\\
&+2h.\gamma_0\gamma'_2\qquad\cos(\ -2g')-2l'.\gamma_0\delta'_2\qquad\sin(\ -2g')\\
&+\tfrac{9}{2}\left[h.\gamma_1\gamma'_3+h'.\delta_1\delta'_3\right]\cos(\ g-3g')-\tfrac{9}{2}\left[l.\delta_1\gamma'_3+l'.\gamma_1\delta'_3\right]\sin(\ g-3g')\\
&+\text{etc.}\qquad\qquad-\text{etc.}\\
&+\tfrac{1}{2}\left[h.\gamma_2\gamma'_1+h'.\delta_2\delta'_1\right]\cos(\ 2g-\ g')-\tfrac{1}{2}\left[l.\delta_2\gamma'_1+l'.\gamma_2\delta'_1\right]\sin(\ 2g-\ g')\\
&+\tfrac{1}{2}\left[h.\gamma_2\gamma'_1+h'.\delta_2\delta'_1\right]\cos(-2g-g')-\tfrac{1}{2}\left[l.\delta_2\gamma'_1-l'.\gamma_2\delta'_1\right]\sin(-2g-g')\\
&+\text{etc.}\qquad\qquad-\text{etc.}
\end{aligned}$$

The numerical value of (H) given by (1) must first be transformed into a series in which both the angles involved are mean anomalies before it can be compared with the value given by the equation just found.

If we find the value of (H) from the preceding equation, it can be checked by means of the tables in BESSEL'S *Werke.*

The expression for $\mu\left(\frac{a}{\Delta}\right)$ is known; and with the expression for (H) just given, we obtain the value of

$$a.\Omega=\mu\left(\frac{a}{\Delta}\right)-(H).$$

The next step is to obtain expressions for the disturbing forces.

Let v the angle between the positive axis of X and the radius-vector measured in the plane of the disturbed body, here called the plane of XY. The differential coefficient of the perturbing function Ω relative to the ordinate Z perpendicular to this plane is found by differentiating Ω relative to z and afterwards putting $z = 0$.
Thus from

$$\Omega = \frac{m'}{1+m}\left[\frac{1}{\varDelta} - \frac{rr'}{r'^3}\,.\,H\right],$$

$$= \frac{m'}{1+m}\left[\frac{1}{\varDelta} - \frac{xx'+yy'+zz'}{r'^3}\right],$$

$$\Delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$

$$= r^2 + r'^2 - 2rr'\,H,$$

we find

$$\frac{d\Omega}{dv} = \frac{m'}{1+m}\left[-\frac{1}{\varDelta^2}\cdot\frac{d\varDelta}{dv} - \frac{r}{r'^2}\cdot\frac{dH}{dv}\right],$$

$$\frac{d\Omega}{dr} = \frac{m'}{1+m}\left[-\frac{1}{\varDelta^2}\left(\frac{r - r'H}{\varDelta}\right) - \frac{H}{r'^2}\right],$$

$$d\Omega = \frac{m'}{1+m}\left[-\frac{1}{\varDelta^2}\,.\,d\Delta - z'\,.\,\frac{dz}{r'^3}\right],$$

$$\Delta\frac{d\varDelta}{dv} = -rr'\frac{dH}{dv}, \qquad \Delta\frac{d\varDelta}{dr} = r - r'H, \qquad \frac{d\varDelta}{dz} = -\frac{z'}{\varDelta}.$$

Hence

$$\frac{d\Omega}{dv} = -\frac{m'}{1+m}\left[\frac{1}{\varDelta^3} - \frac{1}{r'^3}\right] rr'H'$$

$$r\frac{d\Omega}{dr} = \frac{m'}{1+m}\left[\frac{1}{\varDelta^3} - \frac{1}{r'^3}\right] rr'H - \frac{m'}{1+m}\cdot\frac{r^2}{\varDelta^3}$$

$$\frac{d\Omega}{dZ} = -\frac{m'}{1+m}\left[\frac{1}{\varDelta^3} - \frac{1}{r'^3}\right] \sin I\,.\,r' \sin(f' + \Pi')$$

where

$$H' = \sin(f + \Pi)\cos(f' + \Pi') - \cos I \cos(f + \Pi)\sin(f' + \Pi')$$

$$z' = -r'\,.\,\sin I \sin(f' + \Pi').$$

As before the origin of angles here is at the ascending node of the plane of the disturbed body on the plane of the disturbing body, and the plane of reference is that of the disturbed body.

If we differentiate the expressions for $r\frac{d\Omega}{dr}$, $\frac{d\Omega}{dZ}$, we find

$$r^2\frac{d^2\Omega}{dr^2}+r\frac{d\Omega}{dr}=\frac{m'}{1+m}\cdot\frac{3}{\varDelta^5}(r^2-rr'H)^2$$

$$+\frac{m'}{1+m}\left(\frac{1}{\varDelta^3}-\frac{1}{r'^3}\right)rr'H-2\frac{m'}{1+m}\cdot\frac{r^2}{\varDelta^3}$$

$$r\frac{d^2\Omega}{drdZ}=\frac{m'}{1+m}\cdot\frac{3}{\varDelta^5}(r^2-rr'H)\sin I\, r'\sin(f'+\Pi')$$

$$\frac{d^2\Omega}{dZ^2}=\frac{m'}{1+m}\cdot\frac{3}{\varDelta^5}\sin{}^2 I\, r'^2\sin{}^2(f'+\Pi')-\frac{m'}{1+m}\cdot\frac{1}{\varDelta^3}$$

$$\frac{d\Omega}{dZ'}=\frac{m'}{1+m}\left(\frac{1}{\varDelta^3}-\frac{1}{r'^3}\right)\sin I.\,r\sin(f+\Pi)$$

$$r\frac{d^2\Omega}{dr.dZ'}=-\frac{m'}{1+m}\cdot\frac{3}{\varDelta^5}(r^2-rr'H)\sin I.\,r\sin(f+\Pi)+\frac{d\Omega}{dZ'}$$

$$\frac{d^2\Omega}{dZdZ'}=-\frac{m'}{1+m}\cdot\frac{3}{\varDelta^5}\cdot\sin{}^2 I.\,rr'\sin(f+\Pi)\sin(f'+\Pi')+\frac{m'}{1+m}\left(\frac{1}{\varDelta^3}-\frac{1}{r'^3}\right)\cos I$$

To eliminate H from some of these expressions we find from

$$\Delta^2=r^2+r'^2-2rr'.\,H,$$

that

$$\frac{rr'H}{\varDelta^3}=\frac{r^2+r'^2}{2\varDelta^3}-\frac{1}{2\varDelta}$$

The expression for $r\frac{d\Omega}{dr}$ then becomes

$$r\frac{d\Omega}{dr}=\frac{m'}{1+m}\left[\frac{r'^2-r^2}{2\varDelta^3}-\frac{1}{2\varDelta}-\frac{r}{r'^2}H\right]$$

From the value of Δ^2 we have, further,

$$\frac{r^2-rr'H}{\varDelta^5}=-\frac{r'^2-r^2}{2\varDelta^5}+\frac{1}{2\varDelta^3},$$

and hence

$$r\frac{d^2\Omega}{drdZ} = -\frac{3}{2}\frac{m'}{1+m}\left[\frac{r'^2-r^2}{\Delta^5} - \frac{1}{\Delta^3}\right]\sin I . r' \sin(f' + \Pi')$$

$$r\frac{d^2\Omega}{drdZ'} = \frac{3}{2}\frac{m'}{1+m}\left[\frac{r'^2-r^2}{\Delta^5} - \frac{1}{\Delta^3}\right]\sin I . r \sin(f + \Pi) + \frac{d\Omega}{dZ'},$$

the latter of which, by means of the expression for $\frac{d\Omega}{dZ'}$, becomes

$$r\frac{d^2\Omega}{drdZ'} = \frac{3}{2}\frac{m'}{1+m}\left[\frac{r'^2-r^2}{\Delta^5} - \frac{1}{3\Delta^3}\right]\sin I r \sin(f + \Pi) - \frac{m'}{1+m}\sin I\frac{r}{r'^3}\sin(f + \Pi)$$

The expression for Δ^2 also gives

$$\frac{(r^2-rr'H)^2}{\Delta^5} = \frac{(r'^2-r^2)^2}{4\Delta^5} - \frac{r'^2-r^2}{2\Delta^3} + \frac{1}{4\Delta},$$

by means of which we find

$$r^2\frac{d^2\Omega}{dr^2} + r\frac{d\Omega}{dr} = \frac{m'}{1+m}\left[\frac{3(r'^2-r^2)^2}{4\Delta^5} - \frac{r'^2}{\Delta^3} + \frac{1}{4\Delta}\right] - \frac{m'}{1+m}\cdot\frac{r}{r'^2}\cdot H.$$

If we put, for brevity,

$$(I) = \frac{\mu}{a^2}.\sin I\left(\frac{a'}{r'}\right)^2\sin(f' + \Pi')$$

$$(I)' = \frac{\mu}{a^2}.\sin I\left(\frac{a'}{r'}\right)^3.\left(\frac{r}{a}\right)\sin(f + \Pi)$$

$$(I)'' = \frac{\mu}{a^2}.\cos I\left(\frac{a'}{r'}\right)^3$$

the expressions which have been given for the forces, together with the perturbing function, are

$$a\Omega = \mu\left(\frac{a}{\Delta}\right) - (H)$$

$$ar\left(\frac{d\Omega}{dr}\right) = \tfrac{1}{2}\mu a^2\left(\frac{a}{\Delta}\right)^3\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\cdot\frac{r^2}{a^2}\right] - \tfrac{1}{2}\mu\left(\frac{a}{\Delta}\right) - (H)$$

$$a^2\left(\frac{d\Omega}{dZ}\right) = -\mu a^2\left(\frac{a}{\Delta}\right)^3 \cdot \frac{\sin I}{a} \cdot \frac{r'}{a'} \sin(f' + \Pi') + (I)$$

$$ar^2\left(\frac{d^2\Omega}{dr^2}\right) + ar\left(\frac{d\Omega}{dr}\right) = \tfrac{3}{4}\mu a^4\left(\frac{a}{\Delta}\right)^5\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]^2 - \mu a^2\left(\frac{a}{\Delta}\right)^3 \cdot \frac{r'^2}{a'^2} + \tfrac{1}{4}\mu\left(\frac{a}{\Delta}\right) - (H)$$

$$a^2r\left(\frac{d^2\Omega}{drdZ}\right) = -\tfrac{3}{2}\mu a^4\left(\frac{a}{\Delta}\right)^5\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]\frac{\sin I}{a} \cdot \frac{r'}{a'}\sin(f' + \Pi')$$

$$+ \tfrac{3}{2}\mu a^2\left(\frac{a}{\Delta}\right)^3 \frac{\sin I}{a}\frac{r'}{a'}\sin(f' + \Pi')$$

$$a^3\left(\frac{d^2\Omega}{dZ^2}\right) = 3\mu a^4\left(\frac{a}{\Delta}\right)^5 \cdot \frac{\sin^2 I}{a^2} \cdot \frac{r'^2}{a'^2} \cdot \sin^2(f' + \Pi') - \mu\left(\frac{a}{\Delta}\right)^3$$

$$aa'\left(\frac{d\Omega}{dZ'}\right) = \mu a^2\left(\frac{a}{\Delta}\right)^3 \cdot \frac{\sin I}{a}\frac{r}{a}\sin(f + \Pi) - (I)'$$

$$aa'r\left(\frac{d^2\Omega}{drdZ'}\right) = \tfrac{3}{2}\mu a^4\left(\frac{a}{\Delta}\right)^5\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\cdot\frac{r^2}{a^2}\right]\frac{\sin I}{a}\cdot\frac{r}{a}\sin(f + \Pi)$$

$$-\tfrac{1}{2}\mu a^2\left(\frac{a}{\Delta}\right)^3 \cdot \frac{\sin I}{a}\frac{r}{a}\sin(f + \Pi) - (I)'$$

$$a^2a'\left(\frac{d^2\Omega}{dZdZ'}\right) = -3\mu a^4\left(\frac{a}{\Delta}\right)^5 \cdot \frac{\sin^2 I}{a^2} \cdot \frac{r'}{a'}\sin(f' + \Pi')\frac{r}{a}\sin(f + \Pi) + \mu a^2\left(\frac{a}{\Delta}\right)^3\frac{\cos I}{a} - (I)''$$

The form given to these expressions is the one best adapted to numerical computations; and the equations are readily derived from the preceding in which the magnitudes occur in linear form.

Thus from

$$r\,\frac{d\Omega}{dr} = \frac{m'}{1+m}\left[\frac{r'^2 - r^2}{2\Delta^3} - \frac{1}{2\Delta} - \frac{r}{r'^2}H\right]$$

we have

$$ar\frac{d\Omega}{dr} = \frac{\mu}{2}\left[\frac{a^3}{a^2}\cdot\frac{r'^2}{\Delta^3} - \frac{a^3}{a^2}\cdot\frac{r^2}{\Delta^3}\right] - \frac{\mu}{2}\cdot\frac{a}{\Delta} - \frac{\mu}{a'^2}\cdot\frac{a'^2}{r'^2}\cdot\alpha^2\cdot\frac{r}{a}\cdot H$$

$$= \frac{\mu}{2}\left(\frac{a}{\Delta}\right)^3\left[\alpha^2\left(\frac{r'}{a'}\right)^2 - \left(\frac{r}{a}\right)^2\right] - \frac{\mu}{2}\left(\frac{a}{\Delta}\right) - \frac{\mu}{a^2}\left(\frac{a'}{r'}\right)^2\cdot\frac{r}{a}\cdot H$$

$$= \frac{\mu}{2}\left(\frac{a}{\Delta}\right)^3\left[\alpha^2\left(\frac{r'}{a'}\right)^2 - \left(\frac{r}{a}\right)^2\right] - \frac{\mu}{2}\left(\frac{a}{\Delta}\right) - (H),$$

where, as before,

$$\mu = \frac{m'}{1+m}\cdot s, \quad (H) = \frac{\mu}{a^2}\cdot\left(\frac{a'}{r'}\right)^2\cdot\frac{r}{a}\cdot H, \quad \alpha = \frac{a'}{a}$$

In a similar manner all the other expressions for the forces have been derived.

When we compute only perturbations of the first order with respect to the mass we need the perturbing function

$$a\Omega = \mu\left(\frac{a}{\Delta}\right) - H$$

and the forces

$$ar\frac{d\Omega}{dr} = \tfrac{1}{2}\mu\alpha^2\left(\frac{a}{\Delta}\right)^3\left[\frac{r'^2}{a'^2} - \frac{1}{\alpha^2}\cdot\frac{r^2}{a^2}\right] - \tfrac{1}{2}\mu\left(\frac{a}{\Delta}\right) - (H)$$

$$a^2\frac{d\Omega}{dZ} = -\mu\alpha^2\left(\frac{a}{\Delta}\right)^3\cdot\frac{\sin I}{a}\frac{r'}{a'}\sin(f' + \Pi') + (I).$$

The other forces are only needed when we take into the account terms of the second order also with respect to the mass.

An inspection of the expressions for the forces shows that besides the functions

$$\mu\left(\frac{a}{\Delta}\right),\ \mu\alpha^2\left(\frac{a}{\Delta}\right)^3,\ \mu\alpha^4\left(\frac{a}{\Delta}\right)^5$$

we need expressions for the magnitudes

$$\left(\frac{r'}{a'}\right)^2,\ \frac{1}{\alpha^2}\cdot\frac{r^2}{a^2},\ \frac{\sin I}{a}\frac{r'}{a'}\sin(f' + \Pi'),\ \frac{\sin I}{a}\frac{r}{a}\sin(f + \Pi),$$

$$(H),\ (I),\ (I)',\ (I)''.$$

When these are known we multiply the function $\mu a^2 \left(\frac{a}{\Delta}\right)^3$ by

$$\left[\left(\frac{r'}{a'}\right)^2 - \frac{1}{a^2} \cdot \frac{r^2}{a^2}\right], \qquad \frac{\sin I}{a} \cdot \frac{r'}{a'} \sin(f' + \Pi'), \qquad \frac{\sin I}{a} \frac{r}{a} \sin(f + \Pi),$$

$$\left(\frac{r'}{a'}\right)^2, \qquad \frac{\cos I}{a};$$

the function $\mu a \left(\frac{a}{\Delta}\right)^5$ by

$$\frac{3}{4}\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]^2, \qquad \frac{3}{2}\frac{\sin I}{a}\frac{r'}{a'} \sin(f' + \Pi')\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right],$$

$$3\frac{\sin^2 I}{a^2} \cdot \frac{r'^2}{a'^2} \cdot \sin^2(f' + \Pi'), \qquad \frac{3}{2}\frac{\sin I}{a}\frac{r}{a} \sin(f + \Pi)\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]$$

$$3\frac{\sin^2 I}{a^2}\frac{r}{a} \sin(f + \Pi)\frac{r'}{a'} \sin(f' + \Pi').$$

We will now find the expressions for (I), $(I)'$, $(I)''$, and for the various factors just given, that are the most convenient for numerical computation.

We have

$$(I) = \frac{\mu}{a^2} \sin I \left(\frac{a'}{r'}\right)^2 . \sin(f' + \Pi').$$

Putting, for brevity,

$$b = -\frac{\mu}{a^2} \cos\phi' \sin I \cos \Pi'$$

$$b' = \frac{\mu}{a^2} \sin I \sin \Pi',$$

and noting that

$$\left(\frac{a'}{r'}\right)^2 \frac{\sin f'}{\cos\varphi'} = \left[\overset{(0)}{J_{\lambda'}} + \overset{(2)}{J_{\lambda'}}\right] \sin g' + 2\left[\overset{(1)}{J_{2\lambda'}} + \overset{(3)}{J_{2\lambda'}}\right] \sin 2g' + \text{etc.}$$

$$\left(\frac{a'}{r'}\right)^2 \cos f' = \left[\overset{(0)}{J_{\lambda'}} - \overset{(2)}{J_{\lambda'}}\right] \cos g' + 2\left[\overset{(1)}{J_{2\lambda'}} - \overset{(3)}{J_{2\lambda'}}\right] \cos 2g' + \text{etc.}$$

we have

$$\left.\begin{aligned}(I) = b\left[\overset{(0)}{J}_{\lambda'} + \overset{(2)}{J}_{\lambda'}\right]\sin(-\ g') &+ b'\left[\overset{(0)}{J}_{\lambda'} - \overset{(2)}{J}_{\lambda'}\right]\cos(-\ g') \\ + 2b\left[\overset{(1)}{J}_{2\lambda'} + \overset{(3)}{J}_{2\lambda'}\right]\sin(-2g') &+ 2b'\left[\overset{(1)}{J}_{2\lambda'} - \overset{(3)}{J}_{2\lambda'}\right]\cos(-2g') \\ + 3b\left[\overset{(2)}{J}_{3\lambda'} + \overset{(4)}{J}_{3\lambda'}\right]\sin(-3g') &+ 3b'\left[\overset{(2)}{J}_{3\lambda'} - \overset{(4)}{J}_{3\lambda'}\right]\cos(-3g') \\ + \text{etc.} \qquad\qquad &+ \text{etc.}\end{aligned}\right\} \tag{3}$$

The value of $(I)'$ is found from

$$(I)' = \frac{\mu}{a^2}\sin I\left(\frac{a'}{r'}\right)^3 \cdot \frac{r}{a}\sin(f + \Pi).$$

From

$$\frac{r'}{a'} = 1 - e'\cos\varepsilon',$$

we find

$$\left(\frac{a'}{r'}\right)^3 = (1 - e'\cos\varepsilon')^{-3}.$$

Expanding,

$$\begin{aligned}\left(\frac{a'}{r'}\right)^3 = \frac{1}{\cos^3\varphi'} &+ (3e' + \tfrac{27}{8}e'^3 + \text{etc.})\cos g' \\ &+ (\tfrac{9}{2}e'^2 + \tfrac{7}{2}e'^4 + \text{etc.})\cos 2g' \\ &+ \tfrac{53}{8}e'^3\cos 3g' + \tfrac{231}{24}e'^4\cos 4g' + \text{etc.};\end{aligned}$$

which, for brevity, we write,

$$\left(\frac{a'}{r'}\right)^3 = \rho_0 + 2\rho_1\cos g' + 2\rho_2\cos 2g' + 2\rho_3\cos 3g' + \text{etc.}$$

But

$$\frac{r}{a}\cdot\frac{\sin f}{\cos\varphi} = \left[\overset{(0)}{J}_{\lambda} + \overset{(2)}{J}_{\lambda}\right]\sin g + \tfrac{1}{2}\left[\overset{(1)}{J}_{2\lambda} + \overset{(3)}{J}_{2\lambda}\right]\sin 2g + \text{etc.}$$

$$\frac{r}{a}\cdot\cos f = -\tfrac{3}{2}e + \left[\overset{(0)}{J}_{\lambda} - \overset{(2)}{J}_{\lambda}\right]\cos g + \tfrac{1}{2}\left[\overset{(1)}{J}_{2\lambda} - \overset{(3)}{J}_{2\lambda}\right]\cos 2g + \text{etc.}$$

Putting

$$l = \frac{\mu}{a^2} \,.\, \cos\phi \sin I \cos \Pi, \qquad l_1 = \frac{\mu}{a^2} \,.\, \sin I \sin \Pi,$$

$$\gamma_1 = \overset{(0)}{J_\lambda} - \overset{(2)}{J_\lambda} \qquad \delta_1 = \overset{(0)}{J_\lambda} + \overset{(2)}{J_\lambda}$$

$$\gamma_2 = \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda}} - \overset{(3)}{J_{2\lambda}}\right] \qquad \delta_2 = \tfrac{1}{2}\left[\overset{(0)}{J_{2\lambda}} + \overset{(2)}{J_{2\lambda}}\right]$$

etc. etc.,

we have

$$(I)' = \left.\begin{array}{ll}
 & -\frac{3}{2} l_1 e \,.\, \rho_0 \\
+ l \,.\, \rho_0 \,.\, \delta_1 \sin g & + l_1 \,.\, \rho_0 \,.\, \gamma_1 \cos g \\
+ l \,.\, \rho_1 \,.\, \delta_1 \,.\, \sin(\ \ g - \ g') & + l_1 \,.\, \rho_1 \,.\, \gamma_1 \cos(\ \ g - \ g') \\
- l \,.\, \rho_1 \,.\, \delta_1 \,.\, \sin(-g - \ g') & + l_1 \,.\, \rho_1 \gamma_1 \cos(-g - \ g') \\
 & - 2 l_1 e \rho_1 \cos(\ \ \ \ - \ g') \\
+ l \,.\, \rho_2 \,.\, \delta_1 \sin(\ \ g - 2g') & + l_1 \,.\, \rho_2 \,.\, \gamma_1 \cos(\ \ g - 2g') \\
- l \,.\, \rho_2 \,.\, \delta_1 \sin(-g - 2g') & + l_1 \,.\, \rho_2 \,.\, \gamma_1 \cos(-g - 2g') \\
 & - 2 l_1 e \,.\, \rho_2 \cos(\ \ \ \ - 2g') \\
\pm \text{etc.} & \pm \text{etc.}
\end{array}\right\} \qquad (4)$$

For $(I)''$ we have the expression

$$(I)'' = \frac{\mu}{a^2} \,.\, \cos I \left(\frac{a'}{r'}\right)^3.$$

Putting

$l_3 = 2 \,.\, \frac{\mu}{a^2} \cos I,$ and using the ρ_i coefficients as for $(I)'$,

we have

$$(I)'' = \frac{l_3 \,.\, \rho_0}{2} + l_3 \,.\, \rho_1 \cos(-g') + l_3 \,.\, \rho_2 \cos(-2g') + \text{etc.} \qquad (5)$$

To obtain an expression for the factor $\left[\left(\frac{r'}{a'}\right)^2 - \frac{1}{a^2}\frac{r^2}{a^2}\right]$ it is only necessary to have that for $\left(\frac{r}{a}\right)^2$.

In terms of the eccentric anomaly we have, at once,

$$\left(\frac{r}{a}\right)^2 = 1 - 2e\cos\varepsilon + e^2\cos^2\varepsilon$$
$$= 1 + \tfrac{1}{2}e^2 - 2e\cos\varepsilon + \tfrac{1}{2}e^2\cos 2\varepsilon.$$

Substituting the values of $\cos\varepsilon$, and $\cos 2\varepsilon$, we have

$$\left(\frac{r}{a}\right)^2 = 1 + \tfrac{3}{2}e^2 - \tfrac{4}{1}\overset{(1)}{J_\lambda}\cos g - \tfrac{4}{4}\overset{(2)}{J_{2\lambda}}\cos 2g - \tfrac{4}{9}\overset{(3)}{J_{3\lambda}}\cos 3g - \text{etc.}$$

To find an expression for the factor $\frac{\sin I}{a} \cdot \frac{r'}{a'}\sin(f' + \Pi')$, for brevity, we let

$$c_1 = \frac{\sin I}{a} \cdot \cos\phi'\cos\Pi', \qquad c_2 = \frac{\sin I}{a} \cdot \sin\Pi',$$

and from the known expressions for $\frac{r'}{a'}\frac{\sin f'}{\cos\varphi'}$, $\frac{r'}{a'}\cos f'$, we get

$$\begin{aligned}\frac{\sin I}{a}\frac{r'}{a'}\sin(f'+\Pi') = {} & \left[\overset{(0)}{J_{\lambda'}} + \overset{(2)}{J_{\lambda'}}\right]c_1\sin g' + \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}} + \overset{(3)}{J_{2\lambda'}}\right]c_1\sin 2g' + \text{etc.}\\ & - \tfrac{3}{2}e'c_2 + \left[\overset{(0)}{J_{\lambda'}} - \overset{(2)}{J_{\lambda'}}\right]c_2\cos g' + \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda'}} - \overset{(3)}{J_{2\lambda'}}\right]c_2\cos 2g' + \text{etc.}\end{aligned}$$

In the same way, if

$$c_3 = \frac{\sin I}{a} \cdot \cos\phi\cos\Pi, \qquad c_4 = \frac{\sin I}{a} \cdot \sin\Pi,$$

we find

$$\left.\begin{aligned}\frac{\sin I}{a} \cdot \frac{r}{a}\sin(f+\Pi) = {} & \left[\overset{(0)}{J_\lambda} + \overset{(2)}{J_\lambda}\right]c_3\sin g + \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda}} + \overset{(3)}{J_{2\lambda}}\right]c_3\sin 2g + \text{etc.}\\ & - \tfrac{3}{2}ec_4 + \left[\overset{(0)}{J_\lambda} - \overset{(2)}{J_\lambda}\right]c_4\cos g + \tfrac{1}{2}\left[\overset{(1)}{J_{2\lambda}} + \overset{(3)}{J_{2\lambda}}\right]c_4\cos 2g + \text{etc.}\end{aligned}\right\} \quad (6)$$

By means of the expressions for the factors

$$\left(\frac{r}{a}\right)^2, \qquad \frac{\sin I}{a} \cdot \frac{r'}{a'}\sin(f' + \Pi'), \qquad \frac{\sin I}{a} \cdot \frac{r}{a} \cdot \sin(f + \Pi),$$

just given, we can form those for

$$\frac{3}{4}\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]^2$$

$$\frac{3}{2}\frac{\sin I}{a} \cdot \frac{r'}{a'}\sin(f' + \Pi')\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]$$

$$3\frac{\sin^2 I}{a^2} \cdot \frac{r'^2}{a'^2}\sin^2(f' + \Pi')$$

$$\frac{3}{2}\frac{\sin I}{a} \cdot \frac{r}{a}\sin(f + \Pi)\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right]$$

$$3\frac{\sin^2 I}{a^2} \cdot \frac{r}{a}\sin(f + \Pi) \cdot \frac{r'}{a'}\sin(f' + \Pi')$$

CHAPTER IV.

Derivation of the Equations for Determining the Perturbations of the Mean Anomaly, the Radius Vector, and the Latitude, together with Equations for Finding the Values of the Arbitrary Constants of Integration.

HANSEN'S expressions for the general perturbations are

$$n_0 z = n_0 t + g_0 + n_0 \int \left[\overline{W}_0 + \frac{d\overline{W}_0}{dt} . \delta z + \nu^2 \right] dt$$

$$\nu = \qquad C - \tfrac{1}{2} \int \left[\frac{d\overline{W}_0}{dt} + \frac{d^2\overline{W}_0}{dt^2} . \delta z \right] dt$$

$$\frac{dR_0}{dt} = hr \frac{\rho}{a_0} \sin(\omega - \bar{f}) \left(\frac{d\Omega}{dZ}\right) \cos i,$$

where

$$\frac{dW_0}{dt} = h_0 \left\{ 2 \frac{\rho}{r} \cos(\bar{f} - \omega) - 1 + 2 \frac{h^2 \rho}{h_0^2 a_0 \cos^2 \varphi_0} [\cos(\bar{f} - \omega) - 1] \right\} \left(\frac{d\Omega}{dv}\right)$$

$$+ 2h_0 \frac{\rho}{r} \sin(\bar{f} - \omega) r \left(\frac{d\Omega}{dr}\right).$$

In this chapter we will show how these expressions are derived from the equations of motion, and from quantities already known.

The equations for the undisturbed motion of m around the Sun are

$$\frac{d^2x}{dt^2} + k^2(1 + m) \frac{x}{r^3} = 0$$

$$\frac{d^2y}{dt^2} + k^2(1 + m) \frac{y}{r^3} = 0$$

$$\frac{d^2z}{dt^2} + k^2(1 + m) \frac{z}{r^3} = 0$$

The effect of the disturbing action of a body m' on the motion of m around the Sun is given by the expressions

$$m'k^2\left(\frac{x'-x}{\Delta^3} - \frac{x'}{r'^3}\right),\quad m'k^2\left(\frac{y'-y}{\Delta^3} - \frac{y'}{r'^3}\right),\quad m'k^2\left(\frac{z'-z}{\Delta^3} - \frac{z'}{r'^3}\right).$$

Introducing these into the equations given above we have in the case of disturbed motion

$$\begin{aligned}
\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3} &= m'k^2\left(\frac{x'-x}{\Delta^3} - \frac{x'}{r'^3}\right) \\
\frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3} &= m'k^2\left(\frac{y'-y}{\Delta^3} - \frac{y'}{r'^3}\right) \qquad (1) \\
\frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} &= m'k^2\left(\frac{z'-z}{\Delta^3} - \frac{z'}{r'^3}\right)
\end{aligned}$$

The second members of equations (1) show the difference between the action of the body m' on m and on the Sun. The action of any member of bodies m', m'', m''', etc., can be included in the second members of these equations, since the action of all will be similar to that of m'.

The second members can be put in more convenient form if we make use of the function

$$\Omega = \frac{m'}{1+m}\left(\frac{1}{\Delta} - \frac{xx'+yy'+zz'}{r'^3}\right).$$

Differentiating relative to x

$$\frac{d\Omega}{dx} = \frac{m'}{1+m}\left(-\frac{1}{\Delta^2}\cdot\frac{d\Delta}{dx} - \frac{x'}{r'^3}\right).$$

But since

$$\Delta^2 = (x'-x)^2 + (y'-y)^2 + (z'-z)^2,$$

we have

$$\frac{d\Delta}{dx} = -\frac{x'-x}{\Delta};$$

and hence

$$(1+m)\frac{d\Omega}{dx} = m'\left(\frac{x'-x}{\varDelta^3} - \frac{x'}{r'^3}\right).$$

In the same way we derive the partial differential coefficients with respect to y and z.

The equations (1) then become

$$\begin{aligned}
\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3} &= k^2(1+m)\frac{d\Omega}{dx} \\
\frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3} &= k^2(1+m)\frac{d\Omega}{dy} \qquad (2) \\
\frac{d^2z}{dt^2} + k^2(1+m)\frac{z}{r^3} &= k^2(1+m)\frac{d\Omega}{dz}
\end{aligned}$$

Let X, Y, Z, be the disturbing forces represented by the second members of equations (2),

R, the disturbing force in the direction of the disturbed radius-vector,

S, the disturbing force, in the plane of the orbit, perpendicular to the disturbed radius-vector, and positive in the direction of the motion.

If f be the angle between the line of apsides and the radius-vector, the angle between this line and the direction of S will be $90° + f$. We then have

$$X = -S\sin f, \quad Y = S\cos f.$$

In case of R, we have

$$R = X\frac{x}{r} + Y\frac{y}{r};$$

and for S,

$$S = Y\frac{x}{r} - X\frac{y}{r}.$$

From these we find

$$X = R\frac{x}{r} - S\frac{y}{r}$$

$$Y = R\frac{y}{r} + S\frac{x}{r}.$$

If we wish to use polar coördinates we have

$$\frac{d\Omega}{dx} = R\cos f - S\sin f$$

$$\frac{d\Omega}{dy} = R\sin f + S\cos f.$$

From

$$x = r\cos f, \qquad y = r\sin f,$$

we find

$$dx = dr\cos f - rdf\sin f$$
$$dy = dr\sin f + rdf\cos f$$
$$d^2x = d^2r\cos f - rd^2f\sin f - 2dr\,df\sin f - rdf^2\cos f$$
$$d^2y = d^2r\sin f + rd^2f\cos f + 2dr\,df\cos f - rdf^2\sin f$$

From the expressions for dx and dy we find

$$dy\cos f. - dx\sin f = r\,df$$
$$dx\cos f. + dy\sin f = dr,$$

and hence

$$\frac{d\Omega}{dx} = -\frac{1}{r}\cdot\frac{d\Omega}{df}\sin f + \frac{d\Omega}{dr}\cos f$$

$$\frac{d\Omega}{dy} = \frac{1}{r}\cdot\frac{d\Omega}{df}\cos f + \frac{d\Omega}{dr}\sin f;$$

from which we see that

$$R = k^2(1+m)\frac{d\Omega}{dr}, \quad S = k^2(1+m)\frac{1}{r}\frac{d\Omega}{df}.$$

If we multiply the expression for d^2x by $\cos f$, that of d^2y by $\sin f$, and add, we obtain

$$d^2x\cos f + d^2y\sin f = d^2r - r\,df^2.$$

In a similar manner we find

$$d^2y \cos f - d^2x \sin f = r\, d^2f + 2dr\, df.$$

Operating on equations (2) in the same way, we have

$$\frac{d^2x}{dt^2}.\cos f + \frac{d^2y}{dt^2}.\sin f + \frac{k^2(1+m)}{r^2} = X.\cos f + Y.\sin f = R$$

$$\frac{d^2y}{dt^2}.\cos f - \frac{d^2x}{dt^2}.\sin f \qquad = Y.\cos f - X \sin f = S$$

Comparing the two sets of equations, we have

$$\left.\begin{aligned} r\frac{d^2f}{dt^2} + 2\frac{dr}{dt}\,\frac{df}{dt} \qquad &= k^2(1+m)\frac{1}{r}\,\frac{d\Omega}{df} \\ \frac{d^2r}{dt^2} - r\frac{df^2}{dt^2} + \frac{k^2(1+m)}{r^2} &= k^2(1+m)\frac{d\Omega}{dr} \end{aligned}\right\} \qquad (3)$$

The second members of equations (1) and (2) are small, and in a first approximation to the motion of m relative to the Sun, we can neglect them. The integration of equations (2) introduces six arbitrary constants; and the integration of equations (3) introduces four. These constants are the elements which determine the undisturbed motion of m around the Sun. Having these elements, let

a_0 the semi-major axis,
n_0 the mean motion,
g_0 the mean anomaly for the instant $t = 0$,
e_0 the eccentricity,
ϕ_0 the angle of eccentricity,
π_0 the angle between the axis of x and the perihelion,
v_0 the angle between the axis of x and the radius-vector,
f_0 the true anomaly,
ε_0 the eccentric anomaly.

These elements are constants, and give the position of the body for the epoch, or for $t = 0$. Let us now take a system of variable elements, functions of the time, and let them be designated as before, omitting the subscript zero, and writing χ in place

of π_0. The former system may be regarded as the particular values which these elements have at the instant $t = 0$.

In Elliptic motion we have

$$
\begin{aligned}
nt + g_0 &= \varepsilon - e \sin \varepsilon \\
r \cos f &= a \cos \varepsilon - ae \\
r \sin f &= a \cos \phi \sin \varepsilon \\
v &= f + \chi \\
a^3 n^2 &= k^2 (1 + m)
\end{aligned}
$$

Now let $n_0 z$ be the mean anomaly which by means of the constant elements gives the same value for the true longitude that is given by the system of variable elements. Further, let the quantities depending on $n_0 z$ be designated by a superposed dash, and let the true disturbed value of r be given by the relation $r = \bar{r}(1 + \nu)$.

We have then

$$
\begin{aligned}
n_0 z &= \bar{\varepsilon} - e_0 \sin \bar{\varepsilon} \\
\bar{r} \cos \bar{f} &= a_0 \cos \bar{\varepsilon} - a_0 e_0 \\
\bar{r} \sin \bar{f} &= a_0 \cos \phi_0 \sin \bar{\varepsilon} \\
\bar{v} &= \bar{f} + \pi_0 \\
a_0^3 n_0^2 &= k^2 (1 + m).
\end{aligned}
$$

We will now first give Brünnow's method of finding expressions for the perturbation of the time, and of the radius vector.

Neglecting the mass m, multiplying the first of equations (1) by y, the second by x, we have

$$x \frac{dy}{dt} - y \frac{dx}{dt} = \int (Yx - Xy)\, dt + C,$$

C being the constant of integration.

Introducing

$$\cos f = \frac{x}{r}, \text{ and } \sin f = \frac{y}{r},$$

into equations (2), neglecting the mass m, we find

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{k^2.\cos f}{r^2}&=X\\ \frac{d^2y}{dt^2}+\frac{k^2.\sin f}{r^2}&=Y\end{aligned}\right\}\qquad(4)$$

We have also

$$\frac{dx}{dt}=\cos f.\frac{dr}{dt}-r\sin f.\frac{df}{dt}$$

$$\frac{dy}{dt}=\sin f.\frac{dr}{dt}+r\cos f.\frac{df}{dt};$$

and hence

$$x\frac{dy}{dt}-y\frac{dx}{dt}=r^2.\frac{df}{dt},$$

or

$$r^2.\frac{df}{dt}=\int(Yx-Xy)\,dt+C;$$

and

$$r^2.\frac{df}{dt}=\int Sr.\,dt+C.$$

In the undisturbed motion we have

$$r_0^{\,2}.\frac{df_0}{dt}=k\sqrt{p_0},$$

p_0 being the semi-parameter.

Hence

$$r^2\frac{df}{dt}=\int Sr.\,dt+k\sqrt{p_0}$$

$$=\quad k\sqrt{p}.$$

From these relations we derive

$$\frac{\sqrt{p}}{\sqrt{p_0}} = 1 + \frac{1}{k\sqrt{p_0}} \int Sr \,.\, dt, \tag{5}$$

and also

$$\frac{\sqrt{p_0}}{\sqrt{p}} = 1 - \frac{1}{k\sqrt{p_0}} \int \frac{\sqrt{p_0}}{\sqrt{p}}\, Sr \,.\, dt \tag{6}$$

If we eliminate $\frac{1}{r^2}$ from equations (4), noting that

$$r^2 \frac{df}{dt} = k\sqrt{p}, \quad \frac{1}{p} \,.\, \frac{d\sqrt{p}}{dt} = \frac{1}{k} \,.\, \frac{1}{p} \,.\, Sr,$$

we have

$$\begin{aligned} \frac{dx}{dt} + \frac{k \sin f}{\sqrt{p}} &= \int \left[X - \frac{\sin f}{p} \,.\, Sr \right] dt, \\ \frac{dy}{dt} - \frac{k \cos f}{\sqrt{p}} &= \int \left[Y - \frac{\cos f}{p} \,.\, Sr \right] dt, \end{aligned} \tag{7}$$

neglecting the constants of integration.

Since $r = \bar{r}\,(1 + \nu)$, we have also

$$x = \bar{x}\,(1 + \nu), \quad y = \bar{y}\,(1 + \nu).$$

The equations (7) then become

$$\begin{aligned} \bar{x} \,.\, \frac{d\nu}{dt} + (1 + \nu) \frac{d\bar{x}}{dt} + \frac{k \sin f}{\sqrt{p}} &= \int \left(X - \frac{\sin f}{p} \,.\, Sr \right) dt \\ \bar{y} \,.\, \frac{d\nu}{dt} + (1 + \nu) \frac{d\bar{y}}{dt} - \frac{k \cos f}{\sqrt{p}} &= \int \left(Y + \frac{\cos f}{p} \,.\, Sr \right) dt \end{aligned} \tag{8}$$

From the equations

$$\bar{x} = a_0 \cos \bar{\varepsilon} - a_0 e_0, \quad \bar{y} = a_0 \cos \phi_0 \sin \bar{\varepsilon}_0,$$

we have

$$d\bar{x} = -a_0 \sin \bar{\varepsilon}\, d\bar{\varepsilon}$$

$$d\bar{y} = a_0 \cos \phi_0 \,.\, \cos \bar{\varepsilon}\, d\bar{\varepsilon}.$$

Then since

$$dg = \frac{r}{a_0}\, d\varepsilon,\ df = \cos \phi \,.\, \frac{a_0^2}{r^2}\, dg,\ \frac{df}{dz} = \frac{k^2}{hr^2},\ h_0 = \frac{k}{\sqrt{p_0}},$$

using the values of $\sin \bar{\varepsilon}$, $\cos \bar{\varepsilon}$, in terms of $\sin \bar{f}$, $\cos \bar{f}$, we find

$$\frac{d\bar{x}}{dz} = -\frac{k \sin \bar{f}}{\sqrt{p_0}},\ \frac{d\bar{y}}{dz} = \frac{\cos \bar{f} + e_0}{\sqrt{p_0}}.$$

And these give

$$\frac{k \sin \bar{f}}{\sqrt{p}} = -\frac{d\bar{x}}{dz} \,.\, \frac{\sqrt{p_0}}{\sqrt{p}},$$

$$\frac{k \cos \bar{f}}{\sqrt{p}} = \frac{d\bar{y}}{dz} \,.\, \frac{\sqrt{p_0}}{\sqrt{p}} - \frac{ke_0}{\sqrt{p}}$$

$$= \frac{d\bar{y}}{dz} \,.\, \frac{\sqrt{p_0}}{\sqrt{p}} - \frac{ke_0}{\sqrt{p_0}} - \int \frac{e_0}{p} \,.\, Sr\, dt$$

The equations (8) then become

$$\begin{aligned} \bar{x}\frac{d\nu}{dt} + \frac{d\bar{x}}{dz}\left[(1+\nu)\frac{dz}{dt} - \frac{\sqrt{p_0}}{\sqrt{p}}\right] &= \int \left(X - \frac{\sin \bar{f}}{p} \,.\, Sr\right) dt \\ \bar{y}\frac{d\nu}{dt} + \frac{d\bar{y}}{dz}\left[(1+\nu)\frac{dz}{dt} - \frac{\sqrt{p_0}}{\sqrt{p}}\right] &= \int \left(Y + \frac{\cos \bar{f} + e_0}{p} \,.\, Sr\right) dt, \end{aligned} \tag{9}$$

the constant $-\dfrac{ke_0}{\sqrt{p}}$ being included in the integral.

We will now transform equations (9), and for this purpose we multiply the first by $\dfrac{d\bar{y}}{dz}$, the second by $\dfrac{d\bar{x}}{dz}$, and noting that

$$\bar{x}\frac{d\bar{y}}{dz} - \bar{y}\frac{d\bar{x}}{dz} = k\sqrt{p},$$

we have

$$\frac{d\nu}{dt} = \frac{\cos f + e_0}{p_0}\int\left(X - \frac{\sin f}{p} . Sr\right)dt + \frac{\sin f}{p_0}\int\left(Y + \frac{(\cos f + e_0)}{p}\right) Sr\, dt \qquad (10)$$

Now multiply the first of (9) by y, the second by x, putting for $\frac{\sqrt{p_0}}{\sqrt{p}}$ its value given by (6), noting that

$$y\frac{dx}{dt} - x\frac{dy}{dt} = -k\sqrt{p_0},$$

we have

$$(1+\nu)\frac{dz}{dt} = 1 - \frac{1}{k\sqrt{p_0}}\int\frac{p_0}{p}. Sr\,dt - \frac{y}{k\sqrt{p_0}}\int(X - \frac{\sin f}{p}. Sr)dt$$
$$+ \frac{x}{k\sqrt{p_0}}\int(Y + \frac{\cos f + e_0}{p}. Sr)dt \qquad (11)$$

We can write $\frac{dz}{dt}$ in the form

$$\frac{dz}{dt} = 2(1+\nu)\frac{dz}{dt} - (1+\nu)^2 . \frac{dz}{dt} + \nu^2 . \frac{dz}{dt}.$$

We have

$$(1+\nu) = \frac{r}{r}, \quad \frac{df}{dt} = \frac{df}{dz}.\frac{dz}{dt}, \quad \frac{df}{dt} = n\frac{a^2}{r^2}\cos\phi,$$

$$\frac{df}{dt} = n_0 . \frac{a_0^2}{r^2}. \cos\phi_0, \quad a^3 n^2 = a_0^3 n_0^2.$$

Making use of these relations we find

$$\frac{dz}{dt} = \frac{1}{(1+\nu)^2}.\frac{\sqrt{p}}{\sqrt{p_0}};$$

and for $\frac{dz}{dt}$ given above we have

$$\frac{dz}{dt} = 2(1+\nu).\frac{dz}{dt} - \frac{\sqrt{p}}{\sqrt{p_0}} + \frac{\nu^2}{(1+\nu)^2}.\frac{\sqrt{p}}{\sqrt{p_0}}.$$

The equation (11) is thus changed into

$$\frac{dz}{dt} = 1 - \frac{1}{k\sqrt{p_0}}\int\left(1 + 2\frac{\sqrt{p_0}}{\sqrt{p}}\right) Sr\,dt - \frac{2y}{k\sqrt{p_0}}\int\left(X - \frac{\sin f}{p}\,Sr\right)dt$$
$$+ \frac{2x}{k\sqrt{p_0}}\int\left(Y + \frac{\cos f + e_0}{p}\,Sr\right)dt + \frac{\nu^2}{(1+\nu)^2}\cdot\frac{\sqrt{p}}{\sqrt{p_0}}. \tag{12}$$

The equations (10) and (12) can be put in briefer form.

Let

$$X_s = X - \frac{\sin f}{p}\,Sr, \qquad Y_c = Y + \frac{\cos f + e_0}{p}\,Sr.$$

Then

$$\frac{d\nu}{dt} = \frac{\cos f + e_0}{p_0}\int X_s\,dt + \frac{\sin f}{p_8}\int Y_c\,dt,$$
$$\frac{dz}{dt} = 1 - \frac{1}{k\sqrt{p_0}}\int\left(1 + 2\frac{\sqrt{p_0}}{\sqrt{p}}\right) Sr\,dt - \frac{2y}{k\sqrt{p_0}}\int X_s\,dt + \frac{2x}{k\sqrt{p_0}}\int Y_c\,dt \tag{13}$$

The values of x, y, found in these equations we get from

$$x = x_0 + \frac{dx_0}{dt}(z - t) + \tfrac{1}{2}\cdot\frac{d^2x_0}{dt^2}(z - t)^2 + \text{etc.}$$
$$y = y_0 + \frac{dy_0}{dt}(z - t) + \tfrac{1}{2}\cdot\frac{d^2y_0}{dt^2}(z - t)^2 + \text{etc.} \tag{14}$$

From the expressions for $\frac{dx}{dz}$, $\frac{dy}{dz}$, we have also

$$\frac{\cos f + e_0}{p_0} = \frac{1}{k\sqrt{p_0}}\left(\frac{dy_0}{dt} + \tfrac{1}{2}\frac{d^2y_0}{dt^2}(z - t)\right) + \text{etc.}$$
$$-\frac{\sin f}{p_0} = \frac{1}{k\sqrt{p_0}}\left(\frac{dx_0}{dt} + \tfrac{1}{2}\frac{d^2x_0}{dt^2}(z - t)\right) + \text{etc.} \tag{15}$$

The quantities given by equations (14) and (15) are found in equations (13) without the integral sign. They can be put under the sign of integration and regarded

as constant if we designate all magnitudes in these factors dependent on t by a Greek letter.

We thus obtain

$$\begin{aligned}\frac{d(z-t)}{dt} &= -\frac{1}{k\sqrt{p_0}}\int\left(1+2\frac{\sqrt{p_0}}{\sqrt{p}}\right)Sr\,dt - \frac{2}{k\sqrt{p_0}}\int(X_s.\upsilon - Y_c.\xi)\,dt \\ &\quad -\frac{2(z-t)}{k\sqrt{p_0}}\int\left(X_s.\frac{d\upsilon}{d\tau}-Y_c.\frac{d\xi}{d\tau}\right)dt + \nu^2.\frac{p}{p_0} \\ \frac{d\nu}{dt} &= \frac{1}{k\sqrt{p_0}}\int\left(X_s.\frac{d\upsilon}{d\tau}-Y_c.\frac{d\xi}{d\tau}\right)dt + \frac{(z-t)}{k\sqrt{p_0}}\int\left(X_s.\frac{d^2\upsilon}{d\tau^2}-Y_c.\frac{d^2\xi}{d\tau^2}\right)dt\end{aligned} \tag{16}$$

These equations include terms of the second order with respect to the mass. If we put

$$W = -\frac{1}{k\sqrt{p_0}}\int\left(1+\frac{\sqrt{p_0}}{\sqrt{p}}\right).\,Sr\,dt - \frac{2}{k\sqrt{p_0}}\int(X_s.\upsilon - Y_c.\xi)\,dt,$$

we get

$$\left.\begin{aligned} n_0 z &= n_0 t + g_0 + n_0\int\left[W+\frac{dW}{d\tau}.\delta z+\nu^2\right]dt \\ \nu &= N - \tfrac{1}{2}\int\left[\frac{dW}{d\tau}+\frac{d^2W}{d\tau^2}.\delta z\right]dt\end{aligned}\right\} \tag{17}$$

In equations (17) g_0 is the mean anomaly for $t=0$; N is the constant of integration in the value of ν.

From the value of W given above, we have

$$\frac{dW}{dt} = -\frac{1}{k\sqrt{p_0}}\left(1+2\frac{\sqrt{p_0}}{\sqrt{p}}\right)Sr - \frac{2}{k\sqrt{p_0}}(X_s.\upsilon - Y_c.\xi).$$

Now since

$$X = \cos f.\frac{d\Omega}{dr} - \sin f.\frac{1}{r}.\frac{d\Omega}{df}$$

$$Y = \sin f.\frac{d\Omega}{dr} + \cos f.\frac{1}{r}.\frac{d\Omega}{df}$$

$$R = \frac{d\Omega}{dr}$$

$$S = \frac{1}{r}.\frac{d\Omega}{df}$$

neglecting the common factor $k^2(1+m)$,
we have

$$\frac{dW}{dt} = -\frac{1}{k\sqrt{p_0}}\left(1+2\frac{\sqrt{p_0}}{\sqrt{p}}\right)\frac{d\Omega}{df} - \frac{2}{k\sqrt{p_0}}\left(\frac{d\Omega}{dr}\,.\cos f - \frac{1}{r}\,.\sin f\,.\frac{d\Omega}{df}\right)v$$

$$+\frac{2}{k\sqrt{p_0}}\left(\frac{d\Omega}{dr}\sin f + \frac{1}{r}\frac{d\Omega}{df}\cos f\right)\xi + \frac{2}{k\sqrt{p_0}}\left[\frac{\sin f}{p}\,.\frac{d\Omega}{df}\,.\,v + \frac{(\cos f + e_0)}{p}\,.\frac{d\Omega}{df}\,.\,\xi\right].$$

And as

$$v = \rho\sin\omega, \qquad \xi = \rho\cos\omega,$$

this becomes

$$\frac{dW}{dt} = \frac{1}{k\sqrt{p_0}}\Bigg[\left(-1-2\frac{\sqrt{p_0}}{\sqrt{p}}\right)\frac{d\Omega}{df} - 2\rho\sin\omega\,.\cos f\,.\frac{d\Omega}{df} + \frac{2}{r}\,.\,\rho\sin\omega\sin f\,.\frac{d\Omega}{df}$$

$$+2\rho\cos\omega\,.\sin f\,.\frac{d\Omega}{dr} + 2\rho\,.\frac{1}{r}\frac{d\Omega}{df}\cos\omega\cos f + 2\rho\,.\frac{\sin\omega\,.\sin f}{p}\frac{d\Omega}{df}$$

$$+2\rho\frac{\cos\omega\,.\cos f}{p}\frac{d\Omega}{df} + \frac{e_0}{p}\,.\,\rho\cos\omega\frac{d\Omega}{df}\Bigg]$$

$$= \frac{1}{k\sqrt{p_0}}\Bigg[\left(-1-2\frac{\sqrt{p_0}}{\sqrt{p}}\right)\frac{d\Omega}{df} + 2\rho\sin(f-\omega)\frac{d\Omega}{dr} + 2\frac{\rho}{r}\cos(f-\omega)\frac{d\Omega}{df}$$

$$+2\frac{\rho}{p}\cos(f-\omega)\frac{d\Omega}{df} + 2e_0\,.\frac{\rho}{p}\cos\omega\frac{d\Omega}{df}\Bigg]$$

But

$$2e_0\,\rho\cos\omega\,.\frac{h^2}{k^2} - 2p_0\,.\frac{h^2}{k_2} = 2\frac{h^2}{k^2}(e_0\,\rho\cos\omega - p_0) = -\rho\,.\,2\frac{h^2}{k^2};$$

also

$$h_0 = \frac{k}{\sqrt{p_0}}, \qquad h = \frac{k}{\sqrt{p}}.$$

Hence since $k^2(1+m)$ is included in X, Y, R, S, we have

$$\frac{dW}{dt} = h_0\left[2\frac{\rho}{r}\cos(\bar{f}-\omega) - 1 + \frac{2\rho\,.\,h^2}{k^2}(\cos(\bar{f}-\omega)-1)\right]\frac{d\Omega}{d\bar{f}} \tag{18}$$

$$+ 2h_0\rho\,.\sin(\bar{f}-\omega)\frac{d\Omega}{dr}$$

If we write $h_0^2.\,a_0\cos^2\phi_0$ in place of k^2 in equation (18), we have the same expression for $\frac{dW}{dt}$ as that given by HANSEN.

Equations (17) and (18) are fundamental in HANSEN'S method of computing the perturbations. We will now give HANSEN'S method of deriving them.

Using the same notation as before, we have, since

$$\frac{a}{r} = \frac{1+e\cos f}{\cos^2\varphi},$$

also

$$\frac{\bar{r}}{a_0} = \frac{\cos^2\varphi_0}{1+e_0\cos\bar{f}};$$

hence

$$\frac{r.a}{\bar{r}.a_0} = \frac{1+e\cos f}{\cos^2\varphi}\cdot\frac{\cos^2\varphi_0}{1+e_0\cos\bar{f}}.$$

Using $\bar{f}+\pi_0-\chi$ in place of f, and developing, we get

$$\frac{r.a}{\bar{r}.a_0} = \frac{r+\bar{r}\cos\bar{f}.e\cos(\chi-\pi_0)+\bar{r}\sin\bar{f}.e\sin(\chi-\pi_0)}{a_0\cos^2\varphi_0}.$$

Let us put

$$\begin{aligned} e\sin(\chi-\pi_0) &= \eta\cos^2\phi_0, \\ e\cos(\chi-\pi_0) &= \xi\cos^2\phi_0 + e_0; \end{aligned} \tag{19}$$

since $e=\sin\phi$, we have

$$\cos^2\phi = \cos^2\phi_0(1-2e_0\,\xi-\cos^2\phi_0\,\xi^2-\cos^2\phi_0\,\eta^2).$$

With this value of $\cos^2\phi$, and $r = a_0 \cos^2\phi_0 - e_0 r \cos f$,

we find

$$\frac{r.a}{r.a_0} = \frac{a_0 \cos^2\varphi_0 - e_0.\bar{r}\cos\bar{f} + \bar{r}\cos\bar{f}(\xi\cos^2\varphi_0 + e_0) + \bar{r}\sin\bar{f}.\eta\cos^2\varphi_0}{a_0\cos^2\varphi_0}$$

$$= \frac{a_0\cos^2\varphi_0 + \bar{r}\cos\bar{f}.\,\xi\cos^2\varphi_0 + \bar{r}\sin\bar{f}.\,\eta\cos^2\varphi_0}{a_0\cos^2\varphi_0\,(1 - 2e_0\xi - \cos^2\varphi_0\,\xi^2 - \cos^2\varphi_0\eta^2};$$

and hence

$$\frac{r.a}{r.a_0} = \frac{1 + \xi.\frac{\bar{r}}{a_0}.\cos\bar{f} + \eta.\frac{\bar{r}}{a_0}.\sin\bar{f}}{1 - 2e_0\xi - \cos^2\varphi_0\xi^2 - \cos^2\varphi_0\eta^2}.$$

From

$$\frac{dv}{dt} = \frac{df}{dt} = \frac{d\bar{f}}{dz}\cdot\frac{dz}{dt},$$

and

$$\frac{df}{dt} = \frac{k\sqrt{p(1+m)}}{r^2},$$

we have

$$\frac{df}{dt} = n\,.\,\frac{a^2}{r^2}\,.\cos\phi.$$

In like manner we find

$$\frac{d\bar{f}}{dz} = n_0\,.\,\frac{a_0^{\,2}}{\bar{r}^2}\,.\cos\phi_0.$$

We have therefore

$$\frac{dz}{dt} = \frac{n.a^2.\bar{r}^2.\cos\varphi}{n_0.a_0^{\,2}.r^2.\cos\varphi_0}$$

If we put $\frac{n}{n_0} = 1 + b$, substitute the values of $\frac{\bar{r}.a}{r.a_0}$, and $\cos^2\phi$, we get

$$\frac{dz}{dt} = (1 + b)\frac{(1+\xi.\frac{r}{a_0}.\cos\bar{f}+\eta.\frac{\bar{r}}{a_0}\sin\bar{f})^2}{(1-2e_0\xi-\cos^2\varphi_0\xi^2-\cos^2\varphi_0\eta^2)^{\frac{3}{2}}} \tag{20}$$

Further, in the case of ν, we have

$$1 + \nu = \frac{r}{\bar{r}}$$

Then since

$$a^3n^2 = a_0^3n_0^2, \quad \frac{n}{n_0} = (1 + b),$$

and

$$\frac{\cos^2\varphi}{\cos^2\varphi_0} = (1 - 2e_0\,\xi - \cos^2\phi_0\,\xi^2 - \cos^2\phi_0\,\eta^2),$$

we have

$$(1 + \nu) = \frac{1-2e_0\xi-\cos^2\varphi_0.\xi^2-\cos^2\varphi_0.\eta^2}{(1+\frac{r}{a_0}\cos f.\xi+\frac{r}{a_0}\sin f.\eta)^{\frac{2}{3}}(1+b)^{\frac{2}{3}}}.$$

If we let

$$A = 1 + \frac{r}{a_0}\cos f\,.\,\xi + \frac{r}{a_0}.\sin f\,.\,\eta,$$

$$B = 1 - 2e_0\,\xi - \cos^2\phi_0\,\xi^2 - \cos^2\phi_0\,\eta^2,$$

$$\frac{h}{h_0} = \frac{(1+b)^{\frac{1}{3}}}{B^{\frac{1}{2}}},$$

we find

$$\frac{dz}{dt} = (1 + b)\,\frac{A^2}{B^{\frac{3}{2}}}, \quad (1 + \nu) = \frac{B}{A(1+b)^{\frac{2}{3}}}.$$

From the latter we have

$$\left(\frac{\nu}{1+\nu}\right)^2 = 1 - 2(1+b)^{\frac{2}{3}}\frac{A}{B} + (1+b)^{\frac{4}{3}}.\frac{A^2}{B^2}.$$

Hence

$$\left(\frac{\nu}{1+\nu}\right)^2\frac{h_0}{h} = \frac{h_0}{h} - 2(1+b)^{\frac{1}{3}}\frac{A}{B^{\frac{1}{2}}} + (1+b)\frac{A^2}{B^{\frac{3}{2}}},$$

$$= \frac{h_0}{h} - \frac{2h}{h_0}A + \frac{dz}{dt}.$$

If we put

$$W = 2\frac{h}{h_0} - \frac{h_0}{h} - 1 + 2\frac{h}{h_0}.\xi\frac{r}{a_0}\cos f + 2\frac{h}{h_0}.\eta.\frac{r}{a_0}\sin f,$$

we have

$$\frac{dz}{dt} = 1 + W + \frac{h_0}{h}\left(\frac{\nu}{1+\nu}\right)^2. \tag{21}$$

We have yet to express $\frac{h_0}{h}$ in terms of the elements.

From

$$B = 1 - 2e_0\xi - \xi^2.\cos^2\phi_0 - \eta^2.\cos^2\phi_0 = \frac{\cos^2\varphi}{\cos^2\varphi_0},$$

and from

$$\frac{n^2}{n_0^2} = \frac{a_0^3}{a^3},$$

$$1 + b = \frac{n}{n_0},$$

we have

$$\frac{h}{h_0} = \left(\frac{n}{n}\right)^{\frac{1}{3}}.\frac{\cos\varphi_0}{\cos\varphi},$$

or

$$\frac{h}{h_0} = \frac{an}{\cos \varphi} \cdot \frac{\cos \varphi_0}{a_0 n_0} .$$

If we put

$$h_0 = \frac{a_0\, n_0}{\cos \varphi_0},$$

we have

$$h = \frac{an}{\cos \varphi} .$$

These values of h and h_0 being substituted in the expressions for $\overline{W}$, $\frac{dz}{dt}$ is found expressed in terms of the elements and of ν, in a very simple form. To find the relation between $\frac{dz}{dt}$ and ν, we use the equation

$$(1 + \nu)^2 = \frac{B^2}{A^2(1+b)^{\frac{4}{3}}};$$

and as this is also equal to $\frac{h_0}{h.\frac{dz}{dt}}$,

we find

$$\frac{dz}{dt} = \frac{h_0}{h} \cdot \frac{1}{(1+\nu)^2} . \tag{22}$$

For the purpose of keeping the formulæ simple and compact, Hansen makes use of the device of designating the time, and the functions of the time other than the elements, by different letters.

Thus for t, r, ε, f, z, ν, x, y, we write,

τ, ρ, η, ω, ζ, β, ξ, v, respectively.

Whenever we integrate, these new symbols are to be treated as constants, noting that the original symbols are used after integration.

If in equation (21) we introduce τ instead of t we shall have

$$\frac{d\zeta}{d\tau} = 1 + W + \frac{h_0}{h}\left(\frac{\beta}{1+\beta}\right)^2, \tag{23}$$

where

$$W = 2\frac{h}{h_0} - \frac{h_0}{h} - 1 + 2\frac{h}{h_0} \cdot \xi \cdot \frac{\bar{\rho}}{a_0} \cos \bar{\omega} + 2\frac{h}{h_0} \cdot \eta \cdot \frac{\bar{\rho}}{a_0} \sin \bar{\omega} .$$

We have also

$$\frac{d\zeta}{d\tau} = \frac{h_0}{h(1+\beta)^2} . \tag{24}$$

The coördinates of a body vary not only with the time but also with the variable elements. In computations where the elements are assumed constant, that part of the velocity of change in the coördinates arising from variable elements must, evidently, be put equal to zero. Coördinates which have the property of retaining for themselves and for their first differential coefficients the same form in disturbed as in undisturbed motion, HANSEN calls ideal coördinates.

If L be a function of ideal coördinates, it can be expressed as a function of the time and of the constant elements. Thus let the time, as it enters into quantities other than the elements, be itself variable and, as before, designated by τ.

The function dependent on t, τ, and the elements we designate by Λ. Then

$$\frac{dL}{dt} = \overline{\frac{d\Lambda}{d\tau}},$$

or

$$dL = \left(\overline{\frac{d\Lambda}{d\tau}}\right) dt$$

where the superposed dash shows that after differentiation τ is to be changed into t.

Let us write the equation (24) in the form

$$\frac{d\zeta}{d\tau}(1+\beta)^2 = \frac{h_0}{h}.$$

Differentiating relative to τ, we have

$$\frac{d\beta}{d\tau} = -\frac{\frac{d^2\zeta}{d\tau^2}}{2\frac{d\zeta}{d\tau}}(1+\beta).$$

The differentiation of (23) also relative to τ gives

$$\frac{d^2\zeta}{d\tau^2} = \frac{dW}{d\zeta}\cdot\frac{d\zeta}{d\tau} + \frac{h_0}{h}\cdot\frac{2\beta}{(1+\beta)^3}\cdot\frac{d\beta}{d\tau}.$$

Eliminating $\frac{h_0}{h}$ by means of (24), we have

$$\frac{\frac{d^2\zeta}{d\tau^2}}{\frac{d\zeta}{d\tau}} = \frac{dW}{d\zeta} + \frac{2\beta}{1+\beta}\cdot\frac{d\beta}{d\tau}.$$

Substituting in the expression for $\frac{d\beta}{d\tau}$ we have

$$\frac{d\beta}{d\tau} = -\tfrac{1}{2}\cdot\frac{dW}{d\zeta}.$$

Since ν is an ideal coördinate, we get from this

$$\nu = N - \tfrac{1}{2}\int\left(\overline{\frac{dW}{d\zeta}}\right)dt, \tag{25}$$

N being the constant of integration, and the dash having the same signification as before.

This expression for ν is a transformation of that given in the equation

$$1+\nu = \frac{1-2e_0\xi - \cos^2\varphi_0.\xi^2 - \cos^2\varphi_0.\eta^2}{(1+b)^{\frac{3}{2}}(1+\xi.\frac{\bar{r}}{a_0}\cos\bar{f}+\eta.\frac{\bar{r}}{a_0}\sin\bar{f})}.$$

Since z is also an ideal coördinate, we have from (23)

$$n_0 z = n_0 t + g_0 + n_0\int\left\{\overline{W} + \frac{h_0}{h}\left(\frac{\nu}{1+\nu}\right)^2\right\}dt \tag{26}$$

g_0 being the constant of integration and being the mean anomaly for $t = 0$.

When we consider only terms of the first order with respect to the disturbing force, ζ changes into τ, and we have

$$\left.\begin{aligned} n_0 z &= n_0 t + c_0 + n_0 \int \overline{W}_0\, dt \\ \nu &= N - \tfrac{1}{2}\int \left(\frac{\overline{d\,W_0}}{d\tau}\right) dt \end{aligned}\right\} \tag{27}$$

where

$$W_0 = 2\frac{h}{h_0} - \frac{h_0}{h} - 1 + 2\frac{h}{h_0}\,.\,\xi\,.\,\frac{\rho}{a_0}\cos\omega + 2\frac{h}{h_0}\,.\,\eta\,.\,\frac{\rho}{a_0}\sin\omega, \tag{28}$$

and ρ and ω are functions of τ, being found from

$$n_0\,\tau + c_0 = \eta - e_0 \sin\eta$$

$$\rho\cos\omega = a_0\cos\eta - a_0\,e_0$$

$$\rho\sin\omega = a_0\cos\phi_0\sin\eta.$$

Also in the last two terms of W_0, $\frac{h}{h_0}$ is put equal to unity.

When terms of the order of the square and higher powers of the disturbing force are considered, ζ cannot be changed into τ. In this case let

$$n_0\,t = n_0\,\tau + g_0 + n\delta z.$$

Likewise let

$$n_0\,\zeta = n_0\,\tau + g_0 + n\delta\zeta$$

where

$n\delta\zeta$ is a function of τ and t.

According to Taylor's theorem we have

$$W = W_0 + \frac{d\,W_0}{d\tau}\,.\,\delta\zeta + \tfrac{1}{2}\,\frac{d^2 W_0}{d\tau^2}\,.\,\delta\zeta^2 + \text{etc.}$$

the value of W_0 being given by (28).

We then have

$$\frac{dW}{d\zeta} = \frac{dW_0}{d\tau} + \frac{d^2 W_0}{d\tau^2} . \delta\zeta + \tfrac{1}{2} . \frac{d^3 W_0}{d\tau^3} . \delta\zeta^3 + \text{etc.}$$

Retaining only terms of the second order, the equations (25) and (26), replacing $\delta\zeta$ by δz, give

$$\left.\begin{aligned} n_0 z &= n_0 t + g_0 + n_0 \int \left[\overline{W}_0 + \frac{d\overline{W}}{d\tau} . \delta z + \nu^2 \right] dt \\ \nu &= N - \tfrac{1}{2} \int \left[\frac{d\overline{W}_0}{d\tau} + \frac{d^2 \overline{W}_0}{d\tau^2} . \delta z \right] dt \end{aligned}\right\} \tag{29}$$

The equation (26) has been put in simpler form by HILL. For this purpose from (21) and (22) we have

$$\frac{h_0}{h} \left(\frac{\nu}{1+\nu} \right)^2 = \nu^2 \frac{dz}{dt} = \frac{dz}{dt} - (1 + \overline{W}).$$

Hence

$$\nu^2 . \frac{dz}{dt} = \nu^2 \left(\frac{1+\overline{W}}{1-\nu^2} \right).$$

Developing the second member and adding $\overline{W}$, we have

$$n_0 z = n_0 t + g_0 + n_0 \int \frac{\overline{W}+\nu^2}{1-\nu^2} dt. \tag{30}$$

The next step is to express $\frac{dW_0}{dt}$ and $\frac{dh}{dt}$ in terms of the disturbing force. From (19) we find

$$\xi = \frac{e}{\cos^2\varphi_0} . \cos(\chi - \pi_0) - \frac{e_0}{\cos^2\varphi_0}$$

$$\eta = \frac{e}{\cos^2\varphi_0} . \sin(\chi - \pi_0).$$

Using these values of ξ and η, and $e_0 \rho \cos \omega = a_0 \cos^2 \phi_0 - \rho$, in equation (28), we find

$$W_0 = \frac{2\rho}{h_0 a_0 \cos^2 \varphi_0} . h e \cos (\chi - \pi_0 - \omega) + \frac{2\rho}{h_0 a_0 \cos^2 \varphi_0} . h - \frac{h_0}{h} - 1.$$

Since

$$h = \frac{an}{\cos \varphi} = \frac{k\sqrt{1+m}}{\sqrt{p}},$$

we have from the expression of h already given,

$$h = \frac{k^2(1+m)}{r^2 . \frac{dv}{dt}}.$$

By means of

$$f = \bar{f} - \omega - (\chi - \pi_0 - \omega),$$

$$\frac{p}{r} - 1 = e \cos f,$$

$$h = \frac{an}{\cos . \varphi},$$

we may transform the expressions

$$\frac{dv}{dt} = \frac{a^2}{r^2} . n \cos \phi,$$

$$\frac{dr}{dt} = \frac{an}{\cos \varphi} . e \sin f,$$

into

$$r . \frac{dv}{dt} - h = \cos (\bar{f} - \omega) . he \cos (\chi - \pi_0 - \omega) + \sin (\bar{f} - \omega) . he \sin (\chi - \pi_0 - \omega)$$

$$\frac{dr}{dt} = \sin (\bar{f} - \omega) . he \cos (\chi - \pi_0 - \omega) - \cos (\bar{f} - \omega) . he \sin (\chi - \pi_0 - \omega)$$

Multiplying the first of these equations by $\cos(\bar{f}-\omega)$, the second by $\sin(\bar{f}-\omega)$, and adding the results, we have

$$he\cos(\chi-\pi_0-\omega)=\left(r\frac{dv}{dt}-h\right)\cos(\bar{f}-\omega)+\frac{dr}{dt}\sin(\bar{f}-\omega).$$

Substituting this value of $h.e.\cos(\chi-\pi_0-\omega)$ in the expression for W_0, noting that

$$\frac{1}{h_0.a_0.\cos^2\varphi_0}=\frac{h_0}{k^2(1+m)},$$

we have

$$W_0=\frac{2.h_0.\rho}{k^2(1+m)}.\cos(\bar{f}-\omega)\,r\frac{dv}{dt}+\frac{2h_0.\rho}{k^2(1+m)}.\sin(\bar{f}-\omega)\frac{dr}{dt}$$
$$-\frac{2\rho}{h_0.a_0\cos^2\varphi_0}\left[\cos(\bar{f}-\omega)-1\right]h-\frac{h_0}{h}-1.$$

Differentiating relative to the time t alone, τ remaining constant, and having care that all the terms of the expressions be homogeneous, we have

$$\frac{dW_0}{dt}=\frac{2h_0\rho}{k^2(1+m)}.\cos(\bar{f}-\omega)\,r\frac{d^2v}{dt^2}+\frac{2h_0\rho}{k^2(1+m)}.\sin(\bar{f}-\omega).\frac{d^2r}{dt^2}$$
$$-\frac{2\rho}{h_0a_0\cos^2\varphi_0}\left[\cos(\bar{f}-\omega)-1\right]\frac{dh}{dt}+\frac{h_0}{h}.\frac{dh}{dt},$$

and

$$\frac{dh}{dt}=-\frac{k^2(1+m)}{r^2\left(\frac{dv}{dt}\right)^2}.\frac{d^2v}{dt^2}=-\frac{h^2r^2}{k^2(1+m)}\,\frac{d^2v}{dt^2}.$$

Substituting

$$k^2(1+m)\frac{1}{r^2}\left(\frac{d\Omega}{dv}\right)\text{ for }\frac{d^2v}{dt^2},$$

$$k^2(1+m)\left(\frac{d\Omega}{dr}\right)\text{ for }\frac{d^2r}{dt^2},$$

we have

$$\frac{dW_0}{dt} = h_0 \left\{ 2\frac{\rho}{r} \cos(\bar{f}-\omega) - 1 + \frac{2h^2\rho}{h_0 a_0 \cos^2 \varphi_0} [\cos(\bar{f}-\omega) - 1] \right\} \left(\frac{d\Omega}{dv} \right)$$
$$+ 2h_0 \frac{\rho}{r} \sin(f-\omega)\, r \left(\frac{d\Omega}{dr} \right) \tag{30}$$

$$\frac{dh}{dt} = -h^2 \left(\frac{d\Omega}{dv} \right)$$

This expression for $\frac{dW_0}{dt}$ is the one used by HANSEN in his *Auseinandersetzung.* It is given in a much simpler form in his posthumous memoir, and as the latter is the form in which we will employ it, we will now give the process employed by HANSEN to effect the transformation.

Substituting first the value of h, omitting the dash placed over certain quantities, noting that in the posthumous memoir ϕ takes the place of ω, and remembering that we are here concerned only with terms of the first order with respect to the mass, we have

$$\frac{dW}{dt} = \frac{a.n}{\sqrt{1-e^2}} \left\{ 2\frac{\rho}{r} \cos(f-\omega) - 1 + \frac{2\rho}{a(1-e^2)} [\cos(f-\omega) - 1] \right\} \left(\frac{d\Omega}{df} \right)$$
$$+ 2\frac{an}{\sqrt{1-e^2}} \cdot \frac{\rho}{r} \sin(f-\omega)\, r \left(\frac{d\Omega}{dr} \right)$$

From the relation

$$\rho = a(1-e^2) - e\rho \cos \omega$$

we have

$$\frac{\rho}{a(1-e^2)} = 1 - \frac{e\rho \cos \omega}{a(1-e^2)}.$$

An inspection of the value of $\frac{dW}{dt}$ shows that its expression consists of three parts, one independent of τ, the other two multiplied by $\rho \cos \omega$, and $\rho \sin \omega$, respectively.

Put

$$\frac{dW}{dt} = \frac{d\Xi}{dt} + \frac{dY}{dt}\left(\frac{\rho}{a}\cos\omega + \tfrac{3}{2}e\right) + \frac{d\Psi}{dt}\cdot\frac{\rho}{a}\sin\omega,$$

and we have

$$\frac{d\Xi}{n.dt} = -3\frac{a}{\sqrt{1-e^2}}\left\{\left[\frac{ae\cos f}{r} + \frac{e\cos f}{1-e^2} + \frac{1}{1-e^2}\right]\left(\frac{d\Omega}{df}\right) + \frac{ae\sin f}{r}.r\left(\frac{d\Omega}{dr}\right)\right\},$$

$$\frac{dY}{ndt} = 2\frac{a}{\sqrt{1-e^2}}\left\{\left[\frac{a\cos f}{r} + \frac{(\cos f+e)}{1-e^2}\right]\left(\frac{d\Omega}{df}\right) + \frac{a\sin f}{r}.r\left(\frac{d\Omega}{dr}\right)\right\},$$

$$\frac{d\Psi}{ndt} = 2\frac{a}{\sqrt{1-e^2}}\left\{\left[\frac{a\sin f}{r} + \frac{\sin f}{1-e^2}\right]\left(\frac{d\Omega}{df}\right) - \frac{a\cos f}{r}.r\left(\frac{d\Omega}{dr}\right)\right\}.$$

But

$$\frac{df}{dg} = \frac{a^2}{r^2}\sqrt{1-e^2} = \frac{ae\cos f}{r\sqrt{1-e^2}} + \frac{e\cos f}{(1-e^2)^{\frac{3}{2}}} + \frac{1}{(1-e^2)^{\frac{3}{2}}},$$

$$\frac{dr}{dg} = \frac{ae\sin f}{\sqrt{1-e^2}},$$

$$\frac{df}{de} = \left(\frac{a}{r} + \frac{1}{1-e^2}\right)\sin f,$$

$$\frac{dr}{de} = -a\cos f;$$

hence

$$\frac{d\Xi}{ndt} = -3a\left(\frac{d\Omega}{dg}\right),$$

$$\frac{dY}{ndt} = \frac{2}{e}\left[a\left(\frac{d\Omega}{dg}\right) - \frac{1}{\sqrt{1-e^2}}a\left(\frac{d\Omega}{df}\right)\right],$$

$$\frac{d\Psi}{ndt} = \frac{2}{\sqrt{1-e^2}}a\left(\frac{d\Omega}{de}\right).$$

Again from

$$\left(\frac{d\Omega}{dg}\right) = \left(\frac{d\Omega}{df}\right)\left(\frac{df}{dg}\right) + \left(\frac{d\Omega}{dr}\right)\left(\frac{dr}{dg}\right)$$

we have

$$\left(\frac{d\Omega}{df}\right) = \left(\frac{d\Omega}{dg}\right)\frac{r^2}{a^2\sqrt{1-e^2}} - r\left(\frac{d\Omega}{dr}\right)\frac{r\,e\sin f}{a(1-e^2)}.$$

Eliminating $\left(\frac{d\Omega}{df}\right)$ from the expression for $\frac{dY}{ndt}$, we have

$$\frac{dY}{ndt} = \frac{2}{1-e^2}\left\{\frac{a^2(1-e^2)-r^2}{a^2 e}\,a\left(\frac{d\Omega}{dg}\right) + \frac{r\sin f}{a\sqrt{1-e^2}}\,a\,r\left(\frac{d\Omega}{dr}\right)\right.$$

In the same way we find

$$\frac{d\Psi}{ndt} = \frac{2}{1-e^2}\left\{\left[\frac{r}{a}\sin f + \frac{r^2\sin f}{a^2(1-e^2)}\right]a\left(\frac{d\Omega}{dg}\right) - \left[\frac{a\cos f}{r}\sqrt{1-e^2} + \frac{e\sin^2 f}{\sqrt{1-e^2}}\right.\right.$$

$$\left.+\frac{r\,e\sin^2 f}{a(1-e^2)^{\frac{3}{2}}}\right]a\,r\left(\frac{d\Omega}{dr}\right)$$

But if we employ the relation

$$1 = \frac{r}{a(1-e^2)} + \frac{r\,e\cos f}{a(1-e^2)}$$

in the term, $\frac{a\cos f}{r}\sqrt{1-e^2}$, of the preceding expression, the whole term becomes

$$-\left[\frac{r\cos f}{a(1-e^2)^{\frac{3}{2}}} + \frac{e}{\sqrt{1-e^2}} + \frac{r\,e}{a(1-e^2)^{\frac{3}{2}}}\right]a\,r\left(\frac{d\Omega}{dr}\right).$$

Using the equation

$$0 = -r\,e\cos f - r + a(1-e^2),$$

multiplying by

$$\frac{e}{a(1-e^2)^{\frac{3}{2}}}\,a\,r\left(\frac{d\Omega}{dr}\right),$$

adding to the preceding, it becomes

$$-\left[\frac{r\cos f}{a\sqrt{1-e^2}}+\frac{2e}{\sqrt{1-e^2}}\right]ar\left(\frac{d\Omega}{dr}\right).$$

Further, we have

$$\frac{d}{dg}\left[\frac{r}{a}\sin f+\frac{r^2\sin f}{a^2(1-e^2)}\right]=\frac{a}{r}\cos f\sqrt{1-e^2}+\frac{\cos f}{\sqrt{1-e^2}}+\frac{e\sin^2 f}{\sqrt{1-e^2}}+2\frac{r\,e\sin^2 f}{a(1-e^2)^{\frac{3}{2}}}.$$

Reducing this expression in the same manner as employed before, it becomes

$$\frac{d}{dg}\left[\frac{r}{a}\sin f+\frac{r^2\sin f}{a^2(1-e^2)}\right]=\frac{2r\cos f+3ae}{a\sqrt{1-e^2}}.$$

Multiply this by dg, the last expression for $\frac{d\Psi}{ndt}$ becomes

$$\frac{d\Psi}{ndt}=\frac{2}{1-e^2}\left\{\int\frac{2r\cos f+3ae}{a\sqrt{1-e^2}}dg-a\left(\frac{d\Omega}{dg}\right)-\frac{r\cos f+2ae}{a\sqrt{1-e^2}}ar\left(\frac{d\Omega}{ar}\right),\right.$$

the integral to be so taken that it vanishes at the same time with g.

Substituting these values of $\frac{d\Xi}{ndt}, \frac{dY}{ndt}, \frac{d\Psi}{ndt}$, in

$$\frac{dW}{ndt}=\frac{d\Xi}{ndt}+\frac{dY}{ndt}\left(\frac{\rho}{a}\cos\omega+\tfrac{3}{2}e\right)+\frac{d\Psi}{ndt}\frac{\rho}{a}\sin\omega,$$

this expression can be made to take the simple form

$$\frac{dW}{ndt}=Aa\left(\frac{d\Omega}{dg}\right)+Bar\left(\frac{d\Omega}{dr}\right), \tag{31}$$

in which

$$A=-3+\frac{1}{1-e^2}\left\{\left(2\frac{\rho}{a}\cos\omega+3e\right)\frac{a^2(1-e^2)-r^2}{a^2e}+\frac{2\rho\sin\omega}{a\sqrt{1-e^2}}\int\left(\frac{2r}{a}\cos f+3e\right)dg\right\}$$

$$B=\frac{1}{1-e^2}\left\{\left(\frac{2\rho}{a}\cos\omega+3e\right)\frac{r\sin f}{a\sqrt{1-e^2}}-\frac{2\rho\sin\omega}{a\sqrt{1-e^2}}\left(\frac{r}{a}\cos f+2e\right)\right\}.$$

Since

$$\frac{d\,.\,r^2}{a^2\,e\,.\,dg} = 2\,\frac{r\sin f}{a\sqrt{1-e^2}},$$

$$\frac{d\,.\,r^2}{a^2\,e.de} = -2\,\frac{r}{a}\cos f,$$

we have

$$A = -3 + \frac{1}{1-e^2}\left\{\left[\frac{d\,.\,\rho^2}{a^2\,.\,de} - 3e\right]\frac{r^2 - a^2(1-e^2)}{a^2\,e} - \frac{d\,.\,\rho^2}{a^2\,e\,d\gamma}\int\left[\frac{d\,.\,r^2}{a^2\,.\,de} - 3e\right]dg\right\}$$

$$B = \frac{1}{2(1-e^2)}\left\{\frac{d\,.\,\rho^2}{a^2\,e\,.\,d\gamma}\left[\frac{d\,.\,r^2}{a^2\,.\,de} - 4e\right] - \left[\frac{d\,.\,\rho^2}{a^2\,.\,de} - 3e\right]\frac{d\,.\,r^2}{a^2\,e\,.\,dg}\right\}.$$

These expressions for A and B can be much simplified.

Thus from

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - (2e - \tfrac{1}{4}e^3)\cos g - (\tfrac{1}{2}e^2 - \tfrac{1}{6}e^4)\cos 2g - \tfrac{1}{4}e^3\cos 3g - \frac{e^4}{6}\cos 4g - \text{etc.},$$

and a similar expression for $\frac{\rho^2}{a^2}$, we get

$$\frac{d\,.\,\rho^2}{a^2\,e\,.\,d\gamma} = \left(2 - \frac{e^2}{4}\right)\sin\gamma,$$

$$\frac{d\,.\,\rho^2}{a^2\,.\,de} - 3e = -(2 - \tfrac{3}{4}e^2)\cos\gamma,$$

$$\frac{d\,.\,r^2}{a^2\,e\,.\,dg} = \left(2 - \frac{e^2}{4}\right)\sin g + \left(e - \frac{e^3}{3}\right)\sin 2g + \tfrac{3}{4}e^2\sin 3g + \tfrac{2}{3}e^3\sin 4g + \text{etc.},$$

$$\int\left[\frac{d\,.\,r^2}{a^2\,.\,de} - 3e\right]dg = -(2 - \tfrac{3}{4}e^2)\sin g - \left(\frac{e}{2} - \frac{e^3}{3}\right)\sin 2g - \frac{e^2}{4}\sin 3g - \frac{e^3}{6}\sin 4g - \text{etc.},$$

$$\frac{r^2 - a^2(1-e^2)}{a^2\,e} = \tfrac{5}{2}e - \left(2 - \frac{e^2}{4}\right)\cos g - \left(\frac{e}{2} - \frac{e^3}{6}\right)\cos 2g - \frac{e^2}{4}\cos 3g - \frac{e^3}{6}\cos 4g,$$

$$\frac{d\,.\,r^2}{a^2\,.\,de} - 4e = -e - (2 - \tfrac{3}{4}e^2)\cos g - \left(e - \tfrac{2}{3}e^3\right)\cos 2g - \tfrac{3}{4}e^2\cos 3g - \tfrac{2}{3}e^3\cos 4g.$$

From which we obtain

$$
\left.
\begin{array}{ll}
A = -3 + (4 + 2e^2)\cos(\gamma - g) & B = -(2 + e^2)\sin(\gamma - g) \\[1ex]
\quad + \left(e + \frac{e^3}{4}\right)\cos(\gamma - 2g) & \quad -\left(e + \frac{e^3}{4}\right)\sin(\gamma - 2g) \\[1ex]
\quad -\left(5e + \frac{25e^3}{8}\right)\cos\gamma & \quad -\left(e + \frac{7e^3}{8}\right)\sin\gamma \\[1ex]
\quad + \frac{e^2}{2}\cos(\gamma - 3g) & \quad -\frac{3e^2}{4}\sin(\gamma - 3g) \\[1ex]
\quad + \frac{e^3}{3}\cos(\gamma - 4g) & \quad -\frac{2e^3}{3}\sin(\gamma - 4g) \\[1ex]
\quad + \frac{e^3}{24}\cos(\gamma + 2g) & \quad + \frac{e^3}{24}\sin(\gamma + 2g)
\end{array}
\right\} \qquad (32)
$$

These are the expressions of A and B whose values are used in the numerical computations.

When we have the coefficients of the arguments in which γ is $+1$, and -1, we obtain the coefficients of the arguments in which γ is $\pm i$, with very little labor.

Let us resume the expression for $\frac{dW}{ndt}$, that is,

$$\frac{dW}{ndt} = A\,a\left(\frac{d\,\Omega}{dg}\right) + B\,ar\left(\frac{d\,\Omega}{dr}\right)$$

A and B having the values given before.

Since $\frac{r^2}{a^2}$ can be put in the form

$$\frac{r^2}{a^2} = \Sigma R^{(k)}\cos kg,$$

we have

$$\frac{2\,r\sin f}{a\sqrt{1-e^2}} = \frac{d.\frac{r^2}{a^2}}{e.dg} = -\Sigma\frac{k}{e}R^{(k)}\sin kg, \qquad 2\frac{r}{a}\cos f = -\frac{d.\frac{r^2}{a^2}}{de} = -\frac{dR^{(k)}}{de}\cos kg,$$

and

$$\int \left\{ \left(\frac{d.\frac{r^2}{a^2}}{de} \right) - 3\,e \right\} dg = \frac{dR^{(k)}}{kde} \sin kg + \frac{dR^{(0)}}{de}\, g - 3\,eg.$$

But since

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - (2e - \tfrac{1}{4}e^3 + \tfrac{1}{96}e^5) \cos g - (\tfrac{1}{2}e^2 - \tfrac{1}{6}e^4 + \tfrac{1}{48}e^6) \cos 2g$$
$$- (\tfrac{1}{4}e^3 - \tfrac{9}{64}e^5) \cos 3g - \text{etc.}$$

we have

$$\frac{dR^{(0)}}{de} = 3e.$$

Hence the integral just given is simply $\frac{dR^{(k)}}{kde} \sin kg$.

A and B can then be written

$$A = -3 + \frac{1}{1-e^2} \left[\left(2\,\frac{\rho}{a} \cos \omega + 3e \right) \frac{a^2(1-e^2) - r^2}{a^2\, e} - \frac{2\,\rho \sin \omega}{a\sqrt{1-e^2}}\; \frac{dR^{(k)}}{kde} \sin kg \right]$$

$$B = \qquad - \frac{1}{1-e^2} \left[\left(2\,\frac{\rho}{a} \cos \omega + 3e \right) \frac{e}{2}\, k\, R^k \sin kg - \frac{2\,\rho \sin \omega}{a\sqrt{1-e^2}} \left(\frac{dR^{(k)}}{de} \cos kg - 2e \right) \right]$$

Putting

$$\frac{\rho^2}{a^2} = \Sigma\, R^{(\kappa)} \cos \varkappa\, \gamma,$$

we have likewise

$$2\,\frac{\rho}{a} \cos \omega = - \frac{d.\frac{\rho^2}{a^2}}{de} = - \frac{dR^{(\kappa)}}{de} \cos \varkappa\gamma, \quad 2\frac{\rho}{a} \sin \omega = \frac{d.\frac{\rho^2}{a^2}}{e.d\gamma} \sqrt{1-e^2} = - \frac{\varkappa}{e} R^{(\kappa)} \sin \varkappa\gamma.$$

Introducing these values of $2\,\frac{\rho}{a} \cos \omega$, and $2\frac{\rho}{a} \sin \omega$ into the expressions for A and B, after integration relative to γ we can write W in the form

$$W = \alpha^{(\kappa)} \frac{\sin}{\cos} \left(\varkappa\gamma + \beta t \right),$$

where

$$\alpha^{(\varkappa)} = \frac{dR^{(\varkappa)}}{de}\, U + \varkappa\, \frac{R^{(\varkappa)}}{e}\, V$$

$$\beta t = ig + i'g',$$

U and V being two functions depending alone on t.

Putting $\varkappa = +1$, and -1, we have

$$\alpha^{(1)} = \frac{dR^{(1)}}{de}\, U + \frac{R^{(1)}}{e}\, V$$

$$\alpha^{(-1)} = \frac{dR^{(1)}}{de}\, U - \frac{R^{(1)}}{e}\, V;$$

and hence

$$U = \frac{\alpha^{(1)} + \alpha^{(-1)}}{2\,\frac{dR^{(1)}}{de}}\,, \quad V = \frac{\alpha^{(1)} - \alpha^{(-1)}}{2\,\frac{R^{(1)}}{e}}\,.$$

Thus we find

$$\alpha^{(\varkappa)} = \tfrac{1}{2}\left[\frac{\frac{dR^{(\varkappa)}}{de}}{\frac{dR^{(1)}}{de}} + \varkappa\,\frac{R^{(\varkappa)}}{R^{(1)}}\right]\alpha^{(1)} + \tfrac{1}{2}\left[\frac{\frac{dR^{(\varkappa)}}{de}}{\frac{dR^{(1)}}{de}} - \varkappa\,\frac{R^{(\varkappa)}}{R^{(1)}}\right]\alpha^{(-1)};$$

or putting

$$\eta^{(\varkappa)} = \frac{\frac{dR^{(\varkappa)}}{de}}{2\frac{dR^{(1)}}{de}} + \varkappa\,\frac{R^{(\varkappa)}}{2R^{(1)}}$$

$$\theta^{(\varkappa)} = \frac{\frac{dR^{(\varkappa)}}{de}}{2\frac{dR^{(1)}}{de}} - \varkappa\,\frac{R^{(\varkappa)}}{2R^{(1)}},$$

we have

$$\alpha^{(\varkappa)} = \eta^{(\varkappa)}\,\alpha^{(1)} + \theta^{(\varkappa)}\,\alpha^{(-1)}. \tag{33}$$

The values of $\eta^{(\kappa)}$ and $\theta^{(\kappa)}$ are readily found from

$$\frac{\rho^2}{a^2} = 1 + \tfrac{3}{2}\,e^2 - (2\,e - \tfrac{1}{4}\,e^3 + \tfrac{1}{96}\,e^5)\cos\gamma - (\tfrac{1}{2}\,e^2 - \tfrac{1}{6}\,e^4 + \tfrac{1}{48}\,e^6)\cos 2\,\gamma$$
$$- (\tfrac{1}{4}\,e^3 - \tfrac{9}{64}\,e^5)\cos 3\,\gamma - \text{etc.}$$
$$= \Sigma\, R^{(\kappa)} \cos \varkappa\,\gamma.$$

We have

$$R^{(0)} = 1 + \tfrac{3}{2}\,e^2$$
$$R^{(1)} = -(2\,e - \tfrac{1}{4}\,e^3 + \tfrac{1}{96}\,e^5)$$
$$R^{(2)} = -(\tfrac{1}{2}\,e^2 - \tfrac{1}{6}\,e^4 + \tfrac{1}{48}\,e^6)$$
$$R^{(3)} = -(\tfrac{1}{4}\,e^3 - \tfrac{9}{64}\,e^5 \qquad)$$
$$\text{etc.,} = \qquad \text{etc.}$$
$$\frac{d\,R^{(0)}}{de} = 3\,e$$
$$\frac{d\,R^{(1)}}{de} = -(2 - \tfrac{3}{4}\,e^2 + \tfrac{5}{96}\,e^4)$$
$$\frac{d\,R^{(2)}}{de} = -(e - \tfrac{2}{3}\,e^3 + \tfrac{1}{8}\,e^5)$$
$$\frac{d\,R^{(3)}}{de} = -(\tfrac{3}{4}\,e^2 - \tfrac{45}{64}\,e^4)$$
$$\frac{d\,R^{(4)}}{de} = -(\tfrac{2}{3}\,e^3 - \tfrac{4}{5}\,e^5)$$
$$\text{etc.} \qquad = \qquad \text{etc.}$$

For $\eta^{(2)}$ we have

$$\eta^{(2)} = \frac{(e - \frac{2}{3}\,e^3 + \frac{1}{8}\,e^5)}{(4 - \frac{3}{2}\,e^2 + \frac{5}{48}\,e^4)} + \frac{(\frac{1}{2}\,e^2 - \frac{1}{6}\,e^4 + \frac{1}{48}\,e^6)}{(2\,e - \frac{1}{4}\,e^3 + \frac{1}{69}\,e^5)},$$
$$= (\tfrac{1}{4}\,e - \tfrac{7}{96}\,e^3 - \tfrac{1}{192}\,e^5) + (\tfrac{1}{4}\,e - \tfrac{5}{96}\,e^3 + \tfrac{1}{384}\,e^5);$$

or

$$\eta^{(2)} = \tfrac{1}{2}\,e - \tfrac{1}{8}\,e^3 - \tfrac{1}{384}\,e^5. \qquad (34)$$

For $\theta^{(2)}$ we get at once

$$\theta^{(2)} = -\tfrac{1}{48}\,e^3 - \tfrac{1}{128}\,e^5.$$

In a similar way we have

$$\eta^{(3)} = \tfrac{3}{8}\, e^2 - \tfrac{15}{128}\, e^4, \quad \eta^{(4)} = \tfrac{1}{3}\, e^3. \tag{35}$$

In case of the third coördinate we also compute the coefficients of the arguments having no angle γ from those having $\pm\gamma$. For this purpose, putting $\varkappa = 0$ in the expression for $\alpha^{(\kappa)}$ we have

$$\alpha^{(0)} = \frac{d\,R^{(0)}}{de}\, U = \frac{d\,R^{(0)}}{de} \; \frac{\alpha^{(1)} + \alpha^{(-1)}}{2\dfrac{d\,R^{(1)}}{de}} = \eta^{(0)}\,(\alpha^{(1)} + \alpha^{(-1)}),$$

where

$$\eta^{(0)} = \frac{\dfrac{d\,R^{(0)}}{de}}{2\dfrac{d\,R^{(1)}}{de}}.$$

For $\eta^{(0)}$ we then have

$$\eta^{(0)} = -\left(\tfrac{3}{2}\, e + \tfrac{9}{16}\, e^3 \pm \text{etc.}\right). \tag{36}$$

Perturbation of the Third Coördinate.

Let b the angle between the radius-vector and the fundamental plane,
i the inclination of the plane of the orbit to the fundamental plane,
$v - \sigma$ the angular distance from the ascending node to the radius-vector.

We have then

$$\sin b = \sin i \sin (v - \sigma).$$

If we use for i and σ their values for the epoch and call them i_0 and Ω_0, Ω_0 being the longitude of the ascending node, we have

$$\sin b = \sin i_0 \sin (v - \Omega_0) + s;$$

s is the perturbation.

Thus we find

$$s = \sin i \sin (v - \sigma) - \sin i_0 \sin (v - \Omega_0).$$

Putting

$$p = \sin i \sin (\sigma - \Omega_0)\ ,\ q = \sin i \cos (\sigma - \Omega_0) - \sin i_0,$$

we find

$$s = q \sin (v - \Omega_0) - p \cos (v - \Omega_0).$$

Instead of s, let us use

$$u = \frac{r}{a_0}\, s,$$

and we have

$$u = \frac{r}{a_0}\, q \sin (v - \Omega_0) - \frac{r}{a_0}\, p \cos (v - \Omega_0).$$

Introducing τ and calling R the new function taking the place of u, we have, putting $\omega + \pi_0$ for v, π_0 being the longitude of the perihelion,

$$\frac{dR}{dt} = \frac{dq}{dt}\frac{\rho}{a_0} \sin (\omega + \pi_0 - \Omega_0) - \frac{dp}{dt}\frac{\rho}{a_0} \cos (\omega + \pi_0 - \Omega_0).$$

To find $\frac{dq}{dt}$ and $\frac{dp}{dt}$ we will employ the method given by WATSON in the eighth chapter of his *Theoretical Astronomy.*

Thus α and β being direction cosines we have

$$z_1 = \alpha\, x + \beta\, y;$$

also

$$z_1 = r \sin i \sin (v - \sigma).$$

But

$$r = r \cos v, \text{ and } y = r \sin v.$$

Hence

$$z_1 = - x \sin i \sin \sigma + y \sin i \cos \sigma,$$

and

$$\alpha = -\sin i \sin \sigma, \qquad \beta = \sin i \cos \sigma.$$

The values of p and q then are given by the equations

$$p = -\alpha \cos \Omega_0 - \beta \sin \Omega_0,$$

$$q = -\alpha \sin \Omega_0 + \beta \cos \Omega_0 - \sin i_0;$$

from which we have

$$\frac{dp}{dt} = -\cos \Omega_0 \frac{d\alpha}{dt} - \sin \Omega_0 \frac{d\beta}{dt},$$

$$\frac{dq}{dt} = -\sin \Omega_0 \frac{d\alpha}{dt} + \cos \Omega_0 \frac{d\beta}{dt}.$$

From the equation $z_1 = \alpha x + \beta y$ we have, first regarding α and β as constant, then regarding x and y as constant,

$$\left(\frac{dz_1}{dt}\right) = \alpha \frac{dx}{dt} + \beta \frac{dy}{dt}$$

$$\left[\frac{dz_1}{dt}\right] = x \frac{d\alpha}{dt} + y \frac{d\beta}{dt} = 0.$$

Differentiating the first of these, regarding all the quantities variable, we have

$$\frac{d^2z_1}{dt^2} = \frac{d\alpha}{dt}\frac{dx}{dt} + \frac{d\beta}{dt}\frac{dy}{dt} + \alpha \frac{d^2x}{dt^2} + \beta \frac{d^2y}{dt^2}.$$

Z_1 being the component of the disturbing force parallel to the axis z_1, and X and Y the other two components, we have

$$Z_1 = \alpha X + \beta Y + Z \cos i.$$

Writing for X and Y their values

$$\frac{d^2x}{dt^2} + k^2(1+m)\frac{x}{r^3}, \qquad \frac{d^2y}{dt^2} + k^2(1+m)\frac{y}{r^3},$$

and reducing by means of

$$z_1 = \alpha x + \beta y,$$

we have

$$Z_1 = \alpha \frac{d^2 x}{dt^2} + \beta \frac{d^2 y}{dt^2} + k^2 (1 + m) \frac{z_1}{r^3} + Z \cos i,$$

or

$$\frac{d^2 z_1}{dt^2} = \alpha \frac{d^2 x}{dt^2} + \beta \frac{d^2 y}{dt^2} + Z \cos i.$$

Comparing this with the other expression for $\frac{d^2 z_1}{dt^2}$, given above,

we have

$$\frac{d\alpha}{dt}\frac{dx}{dt} + \frac{d\beta}{dt}\frac{dy}{dt} = Z \cos i.$$

From this equation, and the value of $\left[\frac{dz_1}{dt}\right]$, since

$$x \frac{dy}{dt} - y \frac{dx}{dt} = k \sqrt{p (1 + m)} = \frac{1}{h},$$

we find

$$\frac{d\alpha}{dt} = - h\, r \cos i \sin v\, Z,$$

$$\frac{d\beta}{dt} = h\, r \cos i \cos v\, Z.$$

Substituting these values in the expressions for $\frac{dp}{dt}$ and $\frac{dq}{dt}$,

we have

$$\frac{dp}{dt} = h\, r \cos i \sin (v - \Omega_0)\, Z,$$

$$\frac{dq}{dt} = h\, r \cos i \cos (v - \Omega_0)\, Z.$$

Introducing these values into the expression for $\frac{dR}{dt}$

we have

$$\frac{dR}{dt} = h\,r \cos i \cos (v - \Omega_0) \frac{\rho}{a_0} \sin (\omega + \pi_0 - \Omega_0)\, Z$$

$$- h\,r \cos i \sin (v - \Omega_0) \frac{\rho}{a_0} \cos (\omega + \pi_0 - \Omega_0)\, Z$$

$$= h\,r \cos i \frac{\rho}{a_0} \left[\sin \omega \cos \left(v - \Omega_0 - (\pi_0 - \Omega_0)\right)\right] Z$$

$$- h\,r \cos i \frac{\rho}{a_0} \left[\cos \omega \sin \left(v - \Omega_0 - (\pi_0 - \Omega_0)\right)\right] Z$$

$$= h\,r \cos i \frac{\rho}{a_0} \sin (\omega - f) \frac{d\Omega}{dZ}.$$

Introducing $n = \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}$, and $h = \frac{k\sqrt{1+m}}{\sqrt{p}}$,

we have

$$\frac{dR}{ndt} = \frac{1}{\sqrt{1-e^2}} \frac{r}{a} \frac{\rho}{a_0} \sin (\omega - f)\, a^2 \frac{d\Omega}{dZ} \cos i. \tag{37}$$

Let

$$C = \frac{1}{\sqrt{1-e^2}} \frac{r}{a} \frac{\rho}{a_0} \sin (\omega - f);$$

then

$$\frac{dR}{\cos i . ndt} = C . a^2 \left(\frac{d\Omega}{dZ}\right).$$

To find an expression for C similar to those for A and B we have, first,

$$C = \frac{1}{\sqrt{1-e^2}} \left[\frac{\rho}{a_0} \sin \omega . \frac{r}{a} \cos f - \frac{\rho}{a_0} \cos \omega . \frac{r}{a} \sin f\right].$$

Substituting the values of $\frac{r}{a}\cos f$, $\frac{r}{a}\sin f$, given before, and similar ones for $\frac{\rho}{a_0}\cos\omega$, $\frac{\rho}{a_0}\sin\omega$, we find

$$C = \tfrac{1}{4}\left(\frac{d.\rho^2}{a_0^2 de}\right)\left(\frac{d.r^2}{a^2 edg}\right) - \tfrac{1}{4}\left(\frac{d.\rho^2}{a_0^2 edg}\right)\left(\frac{d.r^2}{a^2 de}\right).$$

Substituting the values of these factors we obtain for C the expression

$$\left.\begin{aligned} C = (1 - \tfrac{1}{2}e^2) & \sin(\gamma - g) \\ -(\tfrac{3}{2}e - \tfrac{3}{16}e^3) & \sin\gamma \\ +(\tfrac{1}{2}e - \tfrac{3}{8}e^3) & \sin(\gamma - 2g) \\ + \tfrac{3}{8}e^2 & \sin(\gamma - 3g) \\ + \tfrac{1}{3}e^3 & \sin(\gamma - 4g) \\ - \tfrac{1}{48}e^3 & \sin(\gamma + 2g) \end{aligned}\right\} \qquad (38)$$

Having found the expressions for $\frac{dW}{ndt}$ and $\frac{du}{ndt\,.\cos i}$

we have, finally, for determining the perturbations, the following expressions:

$$n\delta z = n\int \overline{W}\,dt,$$

$$\nu = -\tfrac{1}{2}n\int \frac{d\overline{W}}{d\gamma}\,dt,$$

$$\frac{u}{\cos i} = \int C a^2 \left(\frac{d\Omega}{dZ}\right).$$

Two integrations are needed to find $n\delta z$. We first find W from $\frac{dW}{ndt}$; then, forming $\overline{W}$ and $-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$ from W we have $n\delta z$ and ν by integrating these quantities. In the integration of $\frac{dW}{ndt}$ we give to the constants of integration the form

$$k_0 + k_1\cos\gamma + k_2\sin\gamma + \eta^{(2)}k_1\cos 2\gamma + \eta^{(2)}k_2\sin 2\gamma + \text{etc.}$$

Then in case of $-\frac{1}{2}\frac{dW}{d\gamma}$ we have

$$+\tfrac{1}{2}k_1\sin\gamma-\tfrac{1}{2}k_2\cos\gamma+\eta^{(2)}k_1\sin 2\gamma-\eta^{(2)}k_2\cos 2\gamma+\text{etc.}$$

In the second integration we call the two new constants C and N, and the constants of the results are in the forms

$$C+k_0\,nt+k_1\sin g-k_2\cos g+\tfrac{1}{2}\eta^{(2)}k_1\sin 2g-\tfrac{1}{2}\eta^{(2)}k_2\cos 2g\pm\text{etc.}$$

$$N\qquad -\tfrac{1}{2}k_1\cos g-\tfrac{1}{2}k_2\sin g-\tfrac{1}{2}\eta^{(2)}k_1\cos 2g-\tfrac{1}{2}\eta^{(2)}k_2\sin 2g-\text{etc.}$$

In case of the latitude the constants are given in the form

$$l_0+l_1\sin g+l_2\cos g+\eta^{(2)}l_1\sin 2g+\eta^{(2)}l_2\cos 2g+\text{etc.}$$

The constants are so determined that the perturbations become zero for the epoch of the elements. Hence also the first differential coefficients of the perturbations relative to the time are zero. We substitute the values of g and g' at the epoch in the expressions for $n\delta z$, ν, $\frac{u}{\cos i}$, $\frac{d}{ndt}(n\delta z)$, etc., including in g' the long period term. Putting the constants equal to zero, and designating the values of $n\delta z$, ν, etc., at the epoch by a subscript zero, we have the following equations for determining the values of the constants of integration:

$$C+k_1\sin g-k_2\cos g+\tfrac{1}{2}\eta^{(2)}k_1\sin 2g-\tfrac{1}{2}\eta^{(2)}k_2\cos 2g+\text{etc.}\;+\;(n\delta z)_0=g_0$$

$$k_0+k_1\cos g+k_2\sin g+\eta^{(2)}k_1\cos 2g+\eta^{(2)}k_2\sin 2g+\text{etc.}\;+\;\frac{d}{ndt}(n\delta z)_0=0$$

$$N-\tfrac{1}{2}k_1\cos g-\tfrac{1}{2}k_2\sin g-\tfrac{1}{2}\eta^{(2)}k_1\cos 2g-\tfrac{1}{2}\eta^{(2)}k_2\sin 2g-\text{etc.}\;+\;(\nu)_0=0$$

$$+\tfrac{1}{2}k_1\sin g-\tfrac{1}{2}k_2\cos g+\eta^{(2)}k_1\sin 2g-\eta^{(2)}k_2\cos 2g+\text{etc.}\;+\;\frac{d}{ndt}(\nu)_0=0$$

$$l_0+l_1\sin g+l_2\cos g+\eta^{(2)}l_1\sin 2g+\eta^{(2)}l_2\cos 2g+\text{etc.}\;+\;\left(\frac{u}{\cos i}\right)_0=0$$

$$l_1\cos g-l_2\sin g+\eta^{(2)}l_1\cos 2g-\eta^{(2)}l_2\sin 2g+\text{etc.}\;+\;\frac{d}{ndt}\left(\frac{u}{\cos i}\right)_0=0$$

To find k_1 and k_2, we derive from the preceding

$$k_1 \left[\cos g - e + \eta^{(2)} \cos 2\, g + \eta^{(3)} \cos 3\, g + \text{etc.}\right] + k_2 \left[\sin g + \eta^{(2)} \sin 2\, g + \text{etc.}\right]$$

$$-3\, Z_0 + 6\, (\nu)_0 + 4 \frac{d}{ndt} (n\delta z)_0 = 0$$

$$k_1 \left[\sin g + 2\, \eta^{(2)} \sin 2\, g + 3\, \eta^{(3)} \sin 3\, g + \text{etc.}\right] - k_2 \left[\cos g + 2\, \eta^{(2)} \cos 2\, g + \text{etc.}\right]$$

$$+ 2 \frac{d}{ndt} (\nu)_0 = 0$$

The value of N is found further on.

Having k_1 we find k_0 from

$$- k_0 - e\, k_1 - 3\, Z_0 + 3 \frac{d}{ndt} (n\delta z)_0 + 6\, (\nu)_0 = 0.$$

We have

$$l_0 = - e\, l_2\,,\; N = - \tfrac{2}{3}\, k_0 - \tfrac{e}{6}\, k_1 - \tfrac{1}{2}\, Z_0\,,$$

where Z_0 is the constant of $\overline{W}$.

Let us find the expressions for the constants N and K, K being the constant of integration in the expression for $\delta \frac{h}{h_0}$.

The equation (22) we can put in the form

$$\frac{dz}{dt} = \frac{h_0}{h} - 2\nu + (3\nu^2 - 4\nu^3 \pm \text{etc.}) \frac{h_0}{h} - 2\, \nu \left(\frac{h_0}{h} - 1\right).$$

The differentiation of nz relative to the time gives

$$\frac{dz}{dt} = 1 + k_0 + Z_0 + Z_1 + \text{periodic terms,}$$

where $Z_0 = - 32''.7162$, in the case of Althæa, and Z_1 the part to be added when terms of the second order of the disturbing force are taken into account.

The expression for ν is

$$\nu = N + \text{periodic terms.}$$

The approximate value of $\frac{h_0}{h}$ being 1, the complete expression for the integral of $d\,\frac{h_0}{h}$ is given by

$$\frac{h_0}{h} = 1 + k_3 + \text{periodic terms,}$$

k_3 being the constant of integration.

Putting $(3\nu^2 - 4\nu^3 + \text{etc.})\,\frac{h_0}{h} - 2\nu\left(\frac{h_0}{h} - 1\right) = V_1 +$ periodic terms, and substituting this expression, together with those of ν and $\frac{h_0}{h}$, in the expression for $\frac{dz}{dt}$, we have, preserving only the constant terms,

$$N = \tfrac{1}{2}\,(k_3 - k_0 - Z_0 - Z_1 + V_1).$$

It is necessary now to find the value of k_3 in terms of the constants. If in the expression for $\frac{d\,W_0}{dt}$ given by equation (18) we write for ρ, its equivalent $a_0 \cos{}^2\phi_0 - e_0\,\rho\cos\omega$, we will have

$$d\,W_0 = h_0\left\{2\frac{\rho}{r}\cos(f-\omega) - 1 - 2\frac{h^2}{h_0^2} + 2\frac{h^2}{h_0^2}\frac{\rho\cos(f-\omega)}{a_0\cos{}^2\varphi_0} + 2e_0\frac{h^2}{h_0^2}\frac{\rho\cos\omega}{a_0\cos{}^2\varphi_0}\right\}\left(\frac{d\Omega}{df}\right)dt$$

$$+ 2\,h_0\,\rho\sin(f-\omega)\left(\frac{d\Omega}{dr}\right)dt.$$

We also have

$$d\,\frac{h_0}{h} = h_0\left(\frac{d\Omega}{df}\right)dt.$$

Selecting from the expression for $d\,W_0$ the terms not containing $\rho\cos\omega$ and $\rho\sin\omega$, we have

$$d\,W_0 = -h_0\left(1 + 2\frac{h^2}{h_0^2}\right)\left(\frac{d\Omega}{df}\right)dt.$$

If the eccentric anomaly is taken as the independent variable we have for the complete integral

$$W_0 = k_0 + k_1 \cos \eta + k_2 \sin \eta - h_0 \int \left(1 + 2\frac{h^2}{h_0^2}\right)\left(\frac{d\Omega}{df}\right) dt.$$

Introducing the true anomaly instead of the eccentric, we have,

since $$\cos \eta = \frac{\cos \omega + e}{1 + e \cos \omega}, \quad \sin \eta = \frac{\sin \omega \cos}{1 + e \cos \omega},$$

$$W_0 = k_0 + e_0 k_1 + \frac{k_1}{a_0} \rho \cos \omega + \frac{k_2}{a_0 \cos \varphi_0} \rho \sin \omega - h_0 \int \left(1 + 2\frac{h^2}{h_0^2}\right)\left(\frac{d\Omega}{df}\right) dt.$$

Neglecting the terms having $\rho \cos \omega$ and $\rho \sin \omega$ we have in W_0 the constants k_0 and $e_0\, k_1$.

The integral of $d\,\frac{h_0}{h}$ is

$$\frac{h_0}{h} = 1 + k_3 + h_0 \int \left(\frac{d\Omega}{df}\right) dt.$$

From the expression for $d\,\frac{h_0}{h}$ we find

$$d\,\frac{h}{h_0} = -\frac{h^2}{h_0}\left(\frac{d\Omega}{df}\right) dt.$$

Integrating this, making use of the value of $\frac{h_0}{h}$, and adding the constants, we have

$$2\frac{h}{h_0} - \frac{h_0}{h} = 1 + k_0 + e k_1 - h_0 \int \left(1 + 2\frac{h^2}{h_0^2}\right)\left(\frac{d\Omega}{df}\right) dt.$$

And since the quantities under the sign of integration do not have any constant terms we can write

$$2\frac{h}{h_0} - \frac{h_0}{h} = 1 + k_0 + e k_1 + \text{periodic terms}$$

$$\frac{h_0}{h} = 1 + k_3 \qquad + \text{periodic terms}$$

Since $\left(\frac{h_0}{h}-1\right)$ is a quantity of the order of the disturbing force we have

$$\frac{h}{h_0}=1-\left(\frac{h_0}{h}-1\right)+\left(\frac{h_0}{h}-1\right)^2-\left(\frac{h_0}{h}-1\right)^3\pm\text{etc.},$$

from which we get

$$2\frac{h}{h_0}-\frac{h_0}{h}=4-3\frac{h_0}{h}+2\left(\frac{h_0}{h}-1\right)^2-2\left(\frac{h_0}{h}-1\right)^3\pm\text{etc.}$$

Now putting

$$\left(\frac{h_0}{h}-1\right)^2-\left(\frac{h_0}{h}-1\right)^3\pm\text{etc.}=H_1+\text{periodic terms},$$

substituting this expression and those for

$$2\frac{h}{h_0}-\frac{h_0}{h},\quad \frac{h_0}{h},$$

the preceding expression for

$$2\frac{h}{h_0}-\frac{h_0}{h}$$

gives, preserving only constant terms,

$$k_3=-\tfrac{1}{3}(k_0+ek_1)+\tfrac{2}{3}H_1.$$

Introducing this value of k_3 into the expression for N it becomes

$$N=-\tfrac{1}{6}(4k_0+ek_1+3Z_0)+\tfrac{1}{6}(3V_1+2H_1-3Z_1).$$

Preserving only the terms of the first order we have

$$N=-\tfrac{1}{6}(4k_0+ek_1+3Z_0).$$

To find the value of K, the constant of integration in case of $\delta\frac{h}{h_0}$, we have

$$\frac{h}{h_0}=1+K+\text{periodic terms},$$

also

$$\frac{h_0}{h} = 1 + k_3 + \text{periodic terms.}$$

From these we get

$$\frac{h}{h_0} - 1 + \frac{h_0}{h} - 1 = K + k_3 = H_1.$$

Hence

$$K = -k_3 + H_1 = \tfrac{1}{3}(k_0 + ek_1) + \tfrac{1}{3} H_1;$$

or, neglecting the term of the second order,

$$K = \tfrac{1}{3}(k_0 + ek_1).$$

CHAPTER V.

Numerical Example Giving the Principal Formulæ Needed in the Computation Together with Directions for their Application.

ALTHÆA 119.	JUPITER.
$g = 332^\circ\ 48'\ 53''.2$	$g' = 63\quad 5\quad 48.6$
$\pi = 11\quad 54\quad 21.1$ } 1894.0	$\pi' = 12\quad 36\quad 59.4$ } 1894.0
$\Omega = 203\quad 51\quad 51.5$	$\Omega' = 99\quad 22\quad 59.9$
$i = 5\quad 44\quad 4.6$	$i' = 1\quad 18\quad 36.9$
$\phi = 4\quad 36\quad 24.9$	$\phi' = 2\quad 45\quad 57.2$
$n = 855''.76428$	$n' = 299''.12834$
$\log n = 2.9323542$	$\log n' = 2.4758576$
$\log a = 0.4117683$	$\log a' = 0.7162374$

The epoch is 1894 Aug. 23.0.

The elements of Jupiter are those given by HILL in his *New Theory of Jupiter and Saturn*, in which the epoch is 1850.0. Applying the annual motion of $57''.9032$ in π', of $36''.36617$ in Ω', to HILL's value of π', and of Ω', we have the values given above. The mass of Jupiter is $\frac{1}{1047.879}$. The elements of Althæa are those given in the *Berliner Astronomisches Jahrbuch* for 1896. The ecliptic and mean equinox are for 1890. To reduce from 1890 to 1894 we employ the formulæ of WATSON in his *Theoretical Astronomy*, pp. 100–102.

$$i' = i + \eta \cos(\Omega - \theta)$$

$$\Omega' = \Omega + (t' - t)\frac{dl}{dt} - \eta \sin(\Omega - \theta)\cot . i'$$

$$\pi' = \pi + (t' - t)\frac{dl}{dt} + \eta \sin(\Omega - \theta)\ \text{tau}\ \tfrac{1}{2} i'$$

where

$$\theta = 351^\circ\, 36'\, 10'' + 39''.79\,(t - 1750) - 5''.21\,(t' - t)$$

$$\eta = 0''.468\,(t' - t)$$

$$\frac{dl}{dt} = 50''.246.$$

These expressions for i', ☊$'$ and π', can be used for the disturbed body as well as for the disturbing body by considering the unaccented quantities to be those given, and the accented quantities those whose values are to be found for the time, t'. HARKNESS, in his work, *The Solar Parallax and Its Related Constants*, using the most recent data, gives the following expressions for θ, η, and $\frac{dl}{dt}$, when referred to 1850.0:

$$\theta = 353^\circ\, 34'\, 55'' + 32''.655\,(t - 1850) - 8''.79\,(t' - t),$$

$$\eta = 0''.46654\,(t - 1850),$$

$$\frac{dl}{dt} = \left[50''.23622 + 0''.000220\,(t - 1850)\right](t' - t).$$

Let $\mu = \frac{n}{n'}$,

we have then

$$\mu = 0.34955$$

$$2\mu = 0.69910$$

$$3\mu = 1.04865$$

$$4\mu = 1.39820$$

$$5\mu = 1.74775$$

$$6\mu = 2.09730$$

$$\text{etc.} = \quad \text{etc.}$$

Hence

$$1 - 3\mu = -.04865\,,$$

$$2 - 6\mu = -.09730\,.$$

This shows that the arguments $(g - 3g')$, and $(2g - 6g')$, have coefficients in the final expressions for the perturbations greatly affected by the factors of integration. In case of the argument $(g - 3g')$, we should compute the coefficients with more decimals; also those of $(0 - 3g')$ and $(2g - 3g')$, since in the developments the coefficients of these affect those of $(g - 3g')$.

From

$$\sin \tfrac{1}{2} I . \sin \tfrac{1}{2} (\Psi + \Phi) = \sin \tfrac{1}{2} (\Omega - \Omega') \sin \tfrac{1}{2} (i - i')$$

$$\sin \tfrac{1}{2} I . \cos \tfrac{1}{2} (\Psi + \Phi) = \cos \tfrac{1}{2} (\Omega - \Omega') \sin \tfrac{1}{2} (i - i')$$

$$\cos \tfrac{1}{2} I . \sin \tfrac{1}{2} (\Psi - \Phi) = \sin \tfrac{1}{2} (\Omega - \Omega') \cos \tfrac{1}{2} (i + i')$$

$$\cos \tfrac{1}{2} I . \cos \tfrac{1}{2} (\Psi - \Phi) = \cos \tfrac{1}{2} (\Omega - \Omega') \cos \tfrac{1}{2} (i + i')$$

where, if $\Omega' > \Omega$, we take $\frac{1}{2} (360^\circ + \Omega - \Omega')$, instead of $\frac{1}{2} (\Omega - \Omega')$, we find

$$\begin{aligned} \Psi &= 116^\circ \; 15' \; 36''.7 \\ \Phi &= 11 \;\; 50 \;\; 33.9 \\ I &= 6 \;\; 11 \;\; 35.3 \end{aligned}$$

An independent determination of these quantities is found from the equations

$$\begin{aligned} \cos p \sin q &= \sin i' \cos (\Omega - \Omega') \\ \cos p \cos q &= \cos i' \\ \cos p \sin r &= \cos i' \sin (\Omega - \Omega') \\ \cos p \cos r &= \cos (\Omega - \Omega') \\ \sin p &= \sin i' \sin (\Omega - \Omega') \end{aligned}$$

$$\begin{aligned} \sin I \sin \Phi &= \sin p \\ \sin I \cos \Phi &= \cos p \sin (i - q) \\ \sin I \sin (\Psi - r) &= \sin p \cos (i - q) \\ \sin I \cos (\Psi - r) &= \sin (i - q) \\ \cos I &= \cos p \cos (i - q). \end{aligned}$$

From

$$\Pi = \pi - \Omega - \Phi$$

$$\Pi' = \pi' - \Omega' - \Psi$$

we have

$$\Pi = 156^\circ\ 11'\ 55''.7\ ,\ \Pi' = 156^\circ\ 58'\ 22''.8.$$

Then from

$$k \sin K = \cos I \sin \Pi'$$
$$k \cos K = \cos \Pi'$$
$$k_1 \sin K_1 = \sin \Pi'$$
$$k_1 \cos K_1 = \cos I \cos \Pi'$$

$$p \sin P = 2\alpha^2 \frac{e'}{e} - 2\alpha k \cos (\Pi - K)$$
$$p \cos P = 2\alpha \cos \phi' \, k_1 \sin (\Pi - K_1)$$
$$v \sin V = 2\alpha \cos \phi \, k \sin (\Pi - K)$$
$$v \cos V = 2\alpha \cos \phi \cos \phi' \, k_1 \cos (\Pi - K_1)$$
$$w \sin W = p - 2\alpha^2 \frac{e'}{e} \sin P$$
$$w \cos W = v \cos (V - P)$$
$$w_1 \sin W_1 = v \sin (V - P)$$
$$w_1 \cos W_1 = 2\alpha^2 \frac{e'}{e} \cos P ,$$

we find

$$
\begin{array}{llll}
K = 157^\circ & 5' & 36''.6 & \log k = 9.999614 \\
K_1 = 156 & 51 & 7.4 & \log k_1 = 9.997849 \\
P = 93 & 3 & 27.0 & \log p = 9.932748 \\
V = 359 & 6 & 2.4 & \log v = 0.601463 \\
W = 266 & 4 & 39.5 & \log w = 0.605196 \\
W_1 = 266 & 15 & 38\,0 & \log w_1 = 0.601352
\end{array}
$$

Then from

$$R = 1 + \alpha^2 - 2\alpha^2 e'^2, \quad \gamma_2 = \alpha^2 e'^2,$$

we have

$$\log R = 0\,702855, \quad \log \gamma_2 = 7.976024.$$

The values of the quantities from Π to γ_2 should be found by a duplicate computation without reference to the former computation, since any error in these quantities will affect all that follows.

We now divide the circumference into sixteen parts relative to the mean anomaly, and find the corresponding values of the eccentric anomaly E from

$$g = E - e \sin E,$$

where e is regarded as expressed in seconds of arc. Substituting the sixteen values of e in the equations

$$f \sin (F - P) = w \sin (E - W) - e p$$

$$f \cos (F - P) = w_1 \cos (E + W_1),$$

we obtain the corresponding values of f and F.

Then in a similar manner from

$$Q = F + x$$

$$C = \gamma_0 + \gamma_2 \sin^2 Q$$

$$\log q = \log f + y$$

$$x = s\left(\frac{\gamma_0 \gamma_2}{f^2} + \frac{\gamma_2^2}{2 f^2}\right) \sin 2F + s\left(\frac{3\gamma_0^2 \gamma_2^2}{2 f^4} - \frac{\gamma_2^2}{4 f^2}\right) \sin 4F$$

$$y = \lambda_0 \frac{\gamma_2^2}{4 f^2} - \lambda_0 \left(\frac{\gamma_0 \gamma_2}{f^2} + \frac{\gamma_2^2}{2 f^2}\right) \cos 2F - \lambda_0 \left(\frac{3\gamma_0^2 \gamma_2^2}{2 f^4} - \frac{\gamma_2^2}{4 f^2}\right) \cos 4F,$$

where $s = 206264''.8$, $\log \lambda_0 = 9.63778$,

we find the values of Q, C, $\log q$, x, and y.

Thus we have found all the quantities entering into the expression

$$\left(\frac{\varDelta}{a}\right)^2 = \left(C - q\cos(E' - Q)\right)\left(1 - \frac{\gamma_2}{q}\cos(E' + Q)\right).$$

Instead of this, we use the transformed expression

$$\left(\frac{a}{\varDelta}\right)^n = N^n\left(1 + a^2 - 2a\cos(E' - Q)\right)^{-\frac{n}{2}}\left(1 + b^2 - 2b\cos(E' + Q)\right)^{-\frac{n}{2}},$$

and have, for finding the values of N, a, b^2, the equations

$$\frac{q}{C} = \sin\chi$$

$$\frac{\gamma_2}{q} = \sin\chi_1$$

$$a = tg\,\tfrac{1}{2}\chi$$

$$b = tg\,\tfrac{1}{2}\chi_1$$

$$N = \frac{\sec\frac{1}{2}\chi \sec\frac{1}{2}\chi_1}{\sqrt{C}}.$$

To find the value of $\left(\frac{a}{\Delta}\right)^n$ we put

$$\left(1+a^2-2a\cos(E'-Q)\right)^{-\frac{n}{2}}=\left[\tfrac{1}{2}b_{\frac{n}{2}}^{(0)}+b_{\frac{n}{2}}^{(1)}\cos(E'-Q)+b_{\frac{n}{2}}^{(2)}\cos 2(E'-Q)+\text{etc.}\right]$$

$$\left(1+b^2-2b\cos(E'+Q)\right)^{-\frac{n}{2}}=\left[\tfrac{1}{2}B_{\frac{n}{2}}^{(0)}+B_{\frac{n}{2}}^{(1)}\cos(E'+Q)+B_{\frac{n}{2}}^{(2)}\cos 2(E'+Q)+\text{etc.}\right]$$

For finding the values of the coëfficients in these expressions we use RUNKLE'S *Tables for Determining the Values of the Coefficients in the Perturbative Function of Planetary Motion*, published by the Smithsonian Institution. With the sixteen values of a as arguments we enter these tables and find at once the corresponding values of $b_{\frac{1}{2}}^{(0)}$, then those of $\frac{b_{\frac{1}{2}}^{(1)}}{a}, \frac{b_{\frac{1}{2}}^{(2)}}{a^2}, \frac{b_{\frac{1}{2}}^{(3)}}{a^3}$, etc., etc.; $\frac{a^4}{\beta^4}\cdot b_{\frac{3}{2}}^{(0)}, \frac{a^3}{\beta^4}\cdot b_{\frac{3}{2}}^{(1)}, \frac{a^2}{\beta^4}\cdot b_{\frac{3}{2}}^{(2)}$, etc., etc., where β^4 is found from the sixteen values of $\beta^2=\frac{a^2}{1-a^2}$.

Since b in $\left(1-2b\cos(E'+Q)\right)$ is very small it will suffice to put

$$\tfrac{1}{2}B_{\frac{n}{2}}^{(0)}=1 \quad, B_{\frac{1}{2}}^{(1)}=b$$

$$B_{\frac{3}{2}}^{(1)}=3b,\ B_{\frac{5}{2}}^{(1)}=5b.$$

Then from

$$c_{\frac{n}{2}}^{(i)}=\tfrac{1}{2}N^n B_{\frac{n}{2}}^{(i)}\cos 2iQ$$

$$s_{\frac{n}{2}}^{(i)}=\tfrac{1}{2}N^n B_{\frac{n}{2}}^{(i)}\sin 2iQ,$$

we have, in case of $\mu\left(\frac{a}{\Delta}\right)$,

$$\tfrac{1}{8}c_{\frac{1}{2}}^{(0)}=\tfrac{1}{8}N,\qquad \tfrac{1}{8}c_{\frac{1}{2}}^{(1)}=\tfrac{1}{16}N\,b\cos 2Q,\qquad \tfrac{1}{8}s_{\frac{1}{2}}^{(1)}=\tfrac{1}{16}N\,b\sin 2Q;$$

and, for $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$,

$$\tfrac{1}{8}\overset{(0)}{c_{\frac{3}{2}}} = \tfrac{1}{8}\overset{3}{N}, \quad \tfrac{1}{8}\overset{(1)}{c_{\frac{3}{2}}} = \tfrac{1}{16}\overset{3}{N}\,3b\cos 2Q, \quad \tfrac{1}{8}\overset{(1)}{s_{\frac{3}{2}}} = \tfrac{1}{16}\overset{3}{N}\,3b\sin 2Q.$$

We divide by 8 to save division after quadrature.

With these values of $\overset{(i)}{c_{\frac{n}{2}}}$, $\overset{(i)}{s_{\frac{n}{2}}}$, and the values of the coefficients $\overset{(i)}{b_{\frac{n}{2}}}$, we find the values of k_i, K_i, from

$$k_i \cos K_i = \overset{(i)}{b_{\frac{n}{2}}}\overset{(0)}{c_{\frac{n}{2}}} + \left(\overset{(i-1)}{b_{\frac{n}{2}}} + \overset{(i+1)}{b_{\frac{n}{2}}}\right)\overset{(1)}{c_{\frac{n}{2}}}$$
$$+ \left(\overset{(i-1)}{b_{\frac{n}{2}}} - \overset{(i+1)}{b_{\frac{n}{2}}}\right)\overset{(1)}{s_{\frac{n}{2}}}$$

For $i = 0$, we find k_0 from

$$k_0 = \tfrac{1}{2}\overset{(0)}{b_{\frac{n}{2}}}\overset{(0)}{c_{\frac{n}{2}}} + \overset{(1)}{b_{\frac{n}{2}}}\overset{(1)}{c_{\frac{n}{2}}}.$$

Then in case of $\mu\left(\frac{a}{\Delta}\right)$ from

$$\overset{(c)}{A_{i,\kappa}} = \tfrac{1}{8}m' s k_i \cos\left[i(Q-g) - K_i\right]$$
$$\overset{(s)}{A_{i,\kappa}} = \tfrac{1}{8}m' s k_i \sin\left[i(Q-g) - K_i\right],$$

where m' is the mass of the disturbing body and $s = 206264.''8$;
and from

$$\overset{(c)}{A_{i,\kappa}} = \tfrac{1}{8}m' s \alpha^2 k_i \cos\left[i(Q-g) - K_i\right]$$
$$\overset{(s)}{A_{i,\kappa}} = \tfrac{1}{8}m' s \alpha^2 k_i \sin\left[i(Q-g) - K_i\right],$$

in case of $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$, we find the values of $\overset{(c)}{A_{i,\kappa}}$ and $\overset{(s)}{A_{i,\kappa}}$ for the 16 different points of the circumference, and the various terms of the series.

Again, since $\overset{(c)}{A}_{i,\kappa}$, $\overset{(s)}{A}_{i,\kappa}$ are given in the forms

$$\overset{(c)}{A}_{i,\kappa} = \Sigma \overset{(c)}{C}_{i,\nu} \cos \nu g + \Sigma \overset{(s)}{C}_{i,\nu} \sin \nu g$$

$$\overset{(s)}{A}_{i,\kappa} = \Sigma \overset{(c)}{S}_{i,\nu} \cos \nu g + \Sigma \overset{(s)}{S}_{i,\nu} \sin \nu g,$$

we have the following equations to find the values of the coefficients $\overset{(c)}{C}_{i,\nu}$, $\overset{(s)}{C}_{i,\nu}$, $\overset{(c)}{S}_{i,\nu}$, $\overset{(s)}{S}_{i,\nu}$:

$$\begin{array}{ll}
(0.8\ ) = Y_0 + Y_8 & (\tfrac{0}{8}) = Y_0 - Y_8 \\
(1.9\ ) = Y_1 + Y_9 & (\tfrac{1}{9}) = Y_1 - Y_9 \\
(2.10) = Y_2 + Y_{10} & (\tfrac{2}{10}) = Y_2 - Y_{10} \\
\vdots & \vdots \\
(7.15) = Y_7 + Y_{15} & (\tfrac{7}{15}) = Y_7 - Y_{15}
\end{array}$$

$$\begin{array}{ll}
(0.4) = (0.8\ ) + (4.12) & \\
(1.5) = (1.9\ ) + (5.13) & \\
(2.6) = (2.10) + (6.14) & (0.2) = (0.4) + (2.6) \\
(3.7) = (3.11) + (7.15) & (1.3) = (1.5) + (3.7)
\end{array}$$

$$\begin{aligned}
4\,(c_0 + 2c_8) &= (0.2) \\
4\,(c_0 - 2c_8) &= (1.3) \\
4\,(c_2 + c_6) &= (0.8) - (4.12) \\
4\,(c_2 - c_6) &= \{[(1.9) - (5.13)] - [(3.11) - (7.15)]\} \cos 45^\circ \\
4\,(s_2 + s_6) &= \{[(1.9) - (5.13)] + [(3.11) - (7.15)]\} \cos 45^\circ \\
4\,(s_2 - s_6) &= (2.10) - (6.14)
\end{aligned}$$

$$\begin{aligned}
8\,c_4 &= (0.4) - (2.6) \\
8\,s_4 &= (1.5) - (3.7)
\end{aligned}$$

$$4\,(c_1 + c_7) = \left(\tfrac{0}{8}\right) + \left[\left(\tfrac{2}{10}\right) - \left(\tfrac{6}{14}\right)\right] \cos 45^\circ$$
$$4\,(c_1 - c_7) = \left[\left(\tfrac{1}{9}\right) - \left(\tfrac{7}{15}\right)\right] \cos 22^\circ.5 + \left[\left(\tfrac{3}{11}\right) - \left(\tfrac{5}{13}\right)\right] \cos 67^\circ.5$$
$$4\,(c_3 + c_5) = \left(\tfrac{0}{8}\right) - \left[\left(\tfrac{2}{10}\right) - \left(\tfrac{6}{14}\right)\right] \cos 45^\circ$$
$$4\,(c_3 - c_5) = \left[\left(\tfrac{1}{9}\right) - \left(\tfrac{7}{15}\right)\right] \sin 22^\circ.5 - \left[\left(\tfrac{3}{11}\right) - \left(\tfrac{5}{13}\right)\right] \sin 67^\circ.5$$
$$4\,(s_1 + s_7) = \left[\left(\tfrac{1}{9}\right) + \left(\tfrac{7}{15}\right)\right] \sin 22^\circ.5 + \left[\left(\tfrac{3}{11}\right) + \left(\tfrac{5}{13}\right)\right] \sin 67^\circ.5$$
$$4\,(s_1 - s_7) = \left[\left(\tfrac{2}{10}\right) + \left(\tfrac{6}{14}\right)\right] \cos 45^\circ + \left(\tfrac{4}{12}\right)$$
$$4\,(s_3 + s_5) = \left[\left(\tfrac{1}{9}\right) + \left(\tfrac{7}{15}\right)\right] \cos 22^\circ.5 - \left[\left(\tfrac{3}{11}\right) + \left(\tfrac{5}{13}\right)\right] \cos 67^\circ.5$$
$$4\,(s_3 - s_5) = \left[\left(\tfrac{2}{10}\right) + \left(\tfrac{6}{14}\right)\right] \cos 45^\circ - \left(\tfrac{4}{12}\right)$$

The values of c_ν, s_ν must satisfy the equation

$$\overset{(c)}{A}_{i,\kappa}, \text{ or } \overset{(s)}{A}_{i,\kappa} = \tfrac{1}{2}\, c_0 + c_1 \cos g + c_2 \cos 2g + \text{etc.}$$
$$+\, s_1 \sin g + s_2 \sin 2g + \text{etc.}$$

i answering to i in $\overset{(i)}{b}_{\frac{n}{2}}$, and κ being any one of the numbers, from 0 to 15 inclusive, into which the circumference is divided. We use c_ν, s_ν as abbreviated forms of $\overset{(c)}{C}_{i,\nu}$, $\overset{(s)}{C}_{i,\nu}$, etc. Having found the values of c_ν, s_ν from the 16 different values of $\overset{(c)}{A}_0$, $\overset{(c)}{A}_1$, $\overset{(s)}{A}_1$, $\overset{(c)}{A}_2$, $\overset{(s)}{A}_2$, . . . $\overset{(c)}{A}_9$, $\overset{(s)}{A}_9$, both for $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)$, we have the values of these functions given by the equation

$$\left(\frac{a}{\Delta}\right)^n = \tfrac{1}{2}\Sigma\Sigma\,(\overset{(c)}{C}_{i,\nu} \mp \overset{(s)}{S}_{i,\nu}) \cos\left[(i \mp \nu)\,g - iE'\right] \mp \tfrac{1}{2}\Sigma\Sigma\,(\overset{(s)}{C}_{i,\nu} \pm \overset{(c)}{S}_{i,\nu}) \sin\left[(i \mp \nu) g - iE\right]$$

The values of the most important quantities from the eccentric anomaly E to c_ν, s_ν, needed in the expansion of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$, are given in the following tables, first for $\mu\left(\frac{a}{\Delta}\right)$, and then for $\mu a^2\left(\frac{a}{\Delta}\right)^3$, when not common to both.

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$.

g	E	$E+W$	$E+W_1$	$F-P$	F
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
(0)	0 0 0.0	266 4 39.5	266 15 38.0	266 21 17.2	359 24 44.2
(1)	24 24 4.2	290 28 43.7	290 39 42.2	290 8 7.8	23 11 34.8
(2)	48 26 37.2	314 31 16.7	314 42 15.2	313 40 58.4	46 44 25.4
(3)	71 52 24.9	337 57 4.4	338 8 2.9	336 53 39.3	69 57 6.3
(4)	94 35 14.0	0 39 53.5	0 50 52.0	359 41 1.3	92 44 28.3
(5)	116 36 51.7	22 41 31.2	22 52 29.7	21 59 7.8	115 2 34.8
(6)	138 4 29.4	44 9 8.9	44 20 7.4	43 47 3.8	136 50 30.8
(7)	159 8 19.6	65 12 59.1	65 23 57.6	65 8 48.4	158 12 15.4
(8)	180 0 0.0	86 4 39.5	86 15 38.0	86 13 41.4	179 17 8.4
(9)	200 51 40.4	106 56 19.9	107 7 18.4	107 15 14.8	200 18 41.8
(10)	221 55 30.6	128 0 10.1	128 11 8.6	128 28 47.5	221 32 14.5
(11)	243 23 8.3	149 27 47.8	149 38 46.3	150 8 27.6	243 11 54.6
(12)	265 24 46.0	171 29 25.5	171 40 24.0	172 23 51.4	265 27 18.4
(13)	288 7 35.1	194 12 14.6	194 23 13.1	195 17 19.4	288 20 46.4
(14)	311 33 22.8	217 38 2.3	217 49 0.8	218 43 0.9	311 46 27.9
(15)	335 35 55.8	241 40 35.3	241 51 33.8	242 28 57.5	335 32 24.5
Σ					1613 47 17.9
Σ'					1433 47 18.6

g	Log. f.	y	x	Q	Log. q.	Log. C.
			″	° ′ ″		
(0)	0.612427	—.001251	— 12.2	359 24 32.0	0.611176	0.706582
(1)	0.612078	—.000860	+431.5	23 18 46.3	0.611218	0.706349
(2)	0.609315	—.000081	+598.0	46 54 23.4	0.609234	0.705534
(3)	0.605242	+.000981	+390.0	70 3 36.3	0.606233	0.704403
(4)	0.601312	+.001292	— 58.6	92 43 29.7	0.602604	0.703241
(5)	0.598569	+.000846	—476.9	114 54 37.9	0.599415	0.702241
(6)	0.597310	+.000091	—626.7	136 40 4.1	0.597401	0.701493
(7)	0.597194	—.000956	—435.1	158 5 0.3	0.596238	0.701011
(8)	0.597621	—.001322	— 15.7	179 16 52.7	0.596299	0.700788
(9)	0.598109	—.000997	+408.7	200 25 30.5	0.597112	0.700494
(10)	0.598532	—.000152	+618.1	221 42 32.6	0.598380	0.700021
(11)	0.599177	+.000777	+496.6	243 20 11.2	0.599954	0.699872
(12)	0.600584	+.001278	+ 96.7	265 28 55.1	0.601862	0.700504
(13)	0.603163	+.001032	—363.1	288 14 43.3	0.604195	0.702020
(14)	0.606734	+.000148	—600.1	311 36 27.8	0.606882	0.704038
(15)	0.610302	—.000825	—452.4	335 24 52.1	0.609477	0.705810
Σ	4.823835	+ 3	— 0.5	1613 47 17.4	4.823838	5.622201
Σ'	4.823834	— 2	— 0.7	1433 47 17.9	4.823842	5.622200

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$.

g	χ	χ_1	Log. b.	Log. a.	a.	Log. N.
	° ′ ″	′ ″				
(0)	53 23 45.3	7 57.83	7.063818	9.701484	0.502902	9.695669
(1)	53 26 41.3	7 57.78	7.063792	9.701945	0.503437	9.695880
(2)	53 14 15.6	7 59.97	7.065778	9.699988	0.501173	9.695892
(3)	52 54 33.7	8 3.30	7.068781	9.696876	0.497594	9.695837
(4)	52 28 55.6	8 7.35	7.072405	9.692804	0.492951	9.695616
(5)	52 6 31.2	8 10.95	7.075601	9.689226	0.488907	9.695421
(6)	51 53 41.2	8 13.23	7.077613	9.687169	0.486597	9.695400
(7)	51 46 50.0	8 14.55	7.078774	9.686068	0.485364	9.695430
(8)	51 49 41.2	8 14.49	7.078721	9.686526	0.485877	9.695629
(9)	52 0 52.3	8 13.57	7.077913	9.688321	0.487889	9.696120
(10)	52 18 36.9	8 12.12	7.076635	9.691160	0.491089	9.696905
(11)	52 36 21.2	8 10.34	7.075061	9.693986	0.494294	9.697532
(12)	52 49 37.5	8 8.19	7.073153	9.696093	0.496699	9.697631
(13)	52 58 10.6	8 5.58	7.070825	9.697448	0.498251	9.697141
(14)	53 5 12.5	8 2.58	7.068133	9.698559	0.499527	9.696354
(15)	53 13 54.4	7 59.70	7.065534	9.699932	0.501109	9.695743
Σ				77.553783	3.956815	77.569096
Σ'				77.553803	3.956845	77.569088

g	Log. $\frac{1}{8}c^{(0)}_{\frac{1}{2}}$	Log. $\frac{1}{8}c^{(1)}_{\frac{1}{2}}$	Log. $\frac{1}{8}s^{(1)}_{\frac{1}{2}}$	Log. $b^{(0)}_{\frac{1}{2}}$	Log. $b^{(1)}_{\frac{1}{2}}$	Log. $b^{(2)}_{\frac{1}{2}}$
(0)	8.792579	6.16064	4.47527n	0.332110	9.748094	9.329969
(1)	8.792790	5.98934	6.02920	0.332186	9.748669	9.331018
(2)	8.792802	4.98551n	6.16173	0.331867	9.746235	9.326571
(3)	8.792731	6.05070n	5.97267	0.331369	9.742375	9.319511
(4)	8.792526	6.16734n	5.14693n	0.330730	9.737346	9.310298
(5)	8.792331	5.98219n	6.05562n	0.330182	9.732946	9.302224
(6)	8.792310	4.93934	6.17378n	0.329872	9.730425	9.297590
(7)	8.792340	6.03383	6.01614n	0.329707	9.729076	9.295111
(8)	8.792539	6.17549	4.57507n	0.329776	9.729636	9.296143
(9)	8.793030	6.05359	5.99045	0.320045	9.731836	9.300183
(10)	8.793815	5.23282	6.17067	0.330477	9.735322	9.306586
(11)	8.794442	5.94812n	6.07618	0.330914	9.738805	9.312970
(12)	8.794541	6.16466n	5.36611	0.331246	9.741407	9.317738
(13)	8.794051	6.07296n	5.94202n	0.331460	9.743073	9.320808
(14)	8.793264	5.23742n	6.16200n	0.331637	9.744461	9.323327
(15)	8.792653	5.97789	6.04134n	0.331858	9.746165	9.326443
Σ				2.647715	77.912926	74.508222
Σ'				2.647721	77.912945	74.508268

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$.

g	Log. $\overset{(3)}{b}_{\frac{1}{2}}$	Log. $\overset{(4)}{b}_{\frac{1}{2}}$	Log. $\overset{(5)}{b}_{\frac{1}{2}}$	Log. $\overset{(6)}{b}_{\frac{1}{2}}$	Log. $\overset{(7)}{b}_{\frac{1}{2}}$	Log. $\overset{(8)}{b}_{\frac{1}{2}}$	Log. $\overset{(9)}{b}_{\frac{1}{2}}$
(0)	8.954999	8.60017	8.2570	7.9215	7.5915	7.2654	6.9426
(1)	8.956515	8.60214	8.2594	7.9244	7.5947	7.2691	6.9468
(2)	8.950082	8.59373	8.2490	7.9120	7.5804	7.2528	6.9286
(3)	8.939865	8.58036	8.2326	7.8926	7.5578	7.2271	6.8997
(4)	8.926521	8.56292	8.2110	7.8668	7.5280	7.1932	6.8617
(5)	8.914818	8.54760	8.1921	7.8444	7.5020	7.1636	6.8285
(6)	8.908100	8.53882	8.1812	7.8314	7.4870	7.1466	6.8094
(7)	8.904506	8.53411	8.1754	7.8244	7.4789	7.1373	6.7991
(8)	8.906000	8.53606	8.1778	7.8273	7.4822	7.1411	6.8033
(9)	8.911861	8.54373	8.1872	7.8386	7.4953	7.1561	6.8201
(10)	8.921142	8.55588	8.2024	7.8565	7.5160	7.1796	6.8464
(11)	8.930392	8.56797	8.2172	7.8742	7.5367	7.2031	6.8728
(12)	8.937298	8.57701	8.2285	7.8875	7.5520	7.2205	6.8923
(13)	8.941742	8.58283	8.2355	7.8960	7.5618	7.2317	6.9048
(14)	8.945388	8.58760	8.2415	7.9030	7.5700	7.2410	6.9152
(15)	8.949898	8.59349	8.2488	7.9117	7.5800	7.2524	6.9280
Σ	71.449530	68.55219	65.7484	63.0060	60.3071	57.6402	54.9995
Σ'	71.449597	68.55223	65.7482	63.0063	60.3072	57.6404	54.9998

g	Log. $\frac{1}{8}\overset{3}{N}$	Log. $\frac{1}{8}\overset{(1)}{c}_{\frac{3}{2}}$	Log. $\frac{1}{8}\overset{(1)}{s}_{\frac{3}{2}}$	Log. $\frac{1}{2}\overset{(0)}{b}_{\frac{3}{2}}$	Log. $\overset{(1)}{b}_{\frac{3}{2}}$	Log. $\overset{(2)}{b}_{\frac{3}{2}}$	Log. $\overset{(3)}{b}_{\frac{3}{2}}$
(0)	8.183917	5.42374	3.73837n	0.280319	0.417421	0.200612	9.961097
(1)	8.184550	5.25307	5.29293	0.281000	0.418474	0.202090	9.963016
(2)	8.184586	4.24928n	5.42550	0.278120	0.414013	0.195824	9.954877
(3)	8.184421	5.31430n	5.23627	0.273612	0.406981	0.185917	9.941987
(4)	8.183758	5.43028n	4.40987n	0.267827	0.397890	0.173060	9.925223
(5)	8.183173	5.24454n	5.31797n	0.262860	0.390004	0.161858	9.910585
(6)	8.183110	4.20163	5.43607n	0.260054	0.385513	0.155458	9.902210
(7)	8.183200	5.29621	5.27852n	0.258559	0.383116	0.152039	9.897732
(8)	8.183797	5.43847	3.83805n	0.259184	0.384116	0.153464	9.899598
(9)	8.185270	5.31804	5.25490	0.261621	0.388024	0.159038	9.906900
(10)	8.187625	4.49962	5.43747	0.265530	0.394254	0.167901	9.918485
(11)	8.189506	5.21681n	5.34487	0.269488	0.400515	0.176758	9.930076
(12)	8.189803	5.43364n	4.63509	0.272484	0.405223	0.183435	9.938754
(13)	8.188333	5.34047n	5.20953n	0.274429	0.408267	0.187732	9.944350
(14)	8.185972	4.50257n	5.42714n	0.276036	0.410773	0.191265	9.948948
(15)	8.184139	5.24121	5.30466n	0.278037	0.413885	0.195644	9.954643
Σ	65.482568			2.159554	3.209203	1.421019	79.449192
Σ'	65.482592			2.159606	3.209266	1.421076	79.449289

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$.

g	Log. $b_{\frac{3}{2}}^{(4)}$	Log. $b_{\frac{3}{2}}^{(5)}$	Log. $b_{\frac{3}{2}}^{(6)}$	Log. $b_{\frac{3}{2}}^{(7)}$	Log. $b_{\frac{3}{2}}^{(8)}$	Log. $b_{\frac{3}{2}}^{(9)}$
(0)	9.70884	9.4484	9.1822	8.9118	8.6383	8.3621
(1)	9.71121	9.4512	9.1854	8.9155	8.6423	8.3665
(2)	9.70116	9.4393	9.1716	8.8998	8.6247	8.3471
(3)	9.68524	9.4203	9.1496	8.8747	8.5965	8.3158
(4)	9.66450	9.3955	9.1207	8.8418	8.5595	8.2747
(5)	9.64638	9.3739	9.0956	8.8131	8.5273	8.2389
(6)	9.63600	9.3614	9.0813	8.7968	8.5089	8.2184
(7)	9.63043	9.3549	9.0735	8.7880	8.4991	8.2077
(8)	9.63276	9.3576	9.0766	8.7914	8.5030	8.2119
(9)	9.64181	9.3684	9.0893	8.8058	8.5191	8.2298
(10)	9.65617	9.3856	9.1093	8.8287	8.5449	8.2585
(11)	9.67052	9.4028	9.1292	8.8515	8.5705	8.2868
(12)	9.68125	9.4156	9.1440	8.8684	8.5893	8.3078
(13)	9.68816	9.4237	9.1537	8.8791	8.6015	8.3213
(14)	9.69382	9.4305	9.1614	8.8882	8.6118	8.3329
(15)	9.70087	9.4389	9.1711	8.8992	8.6240	8.3464
Σ	77.37450	75.2339	73.0471	70.8269	68.5804	66.3134
Σ'	77.37462	75.2341	73.0474	70.8269	68.5803	66.3132

g	Log. k_0	Log. k_1	Log. k_2	Log. k_3	Log. k_4	Log. k_5	Log. k_6	Log. k_7
(0)	8.824187	8.54492	8.12562	7.750420	7.39550	7.0523	6.7168	6.4105
(1)	8.824302	8.54433	8.12588	7.751220	7.39678	7.0540	6.7190	6.4054
(2)	8.823605	8.53875	8.11916	7.742693	7.38634	7.0416	6.7046	6.3714
(3)	8.822665	8.53172	8.10982	7.730361	7.37091	7.0232	6.6832	6.3298
(4)	8.821701	8.52543	8.09963	7.716100	7.35261	7.0007	6.6565	6.2932
(5)	8.821143	8.52236	8.09246	7.705215	7.33807	6.9826	6.6349	6.2764
(6)	8.821183	8.52360	8.09009	7.700585	7.33130	6.9737	6.6239	6.2809
(7)	8.821397	8.52470	8.08981	7.699023	7.32855	6.9698	6.6187	6.2913
(8)	8.821810	8.52671	8.09164	7.701551	7.33151	6.9732	6.6226	6.3027
(9)	8.822444	8.52829	8.09567	7.707159	7.33895	6.9824	6.6337	6.3093
(10)	8.823323	8.52965	8.10077	7.715298	7.35002	6.9965	6.6506	6.3129
(11)	8.824009	8.53059	8.10550	7.723069	7.36070	7.0100	6.6669	6.3147
(12)	8.824233	8.53159	8.10915	7.728940	7.36874	7.0202	6.6793	6.3196
(13)	8.824055	8.53359	8.11233	7.733450	7.37462	7.0274	6.6879	6.3342
(14)	8.823809	8.53721	8.11622	7.738311	7.38053	7.0345	6.6960	6.3608
(15)	8.823826	8.54164	8.12113	7.744423	7.38795	7.0433	6.7062	6.3901
Σ	70.583851	68.25726	64.85258	61.793910	58.89655	56.0927	53.3503	50.6520
Σ'	70.583841	68.25722	64.85260	61.793920	58.89653	56.0926	53.3505	50.6512

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$.

g	Log. k_8	Log. k_9	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_9
			′	′	′	′	′	′	′	
(0)	6.0606	5.7378	— 0.6	— 0.4	— 0.3	— 0.3	— 0.3	— 0.3	— 0.3	— 0.3
(1)	6.0636	5.7413	+20.3	+12.9	+11.4	+11.1	+10.6	+10.1	+ 9.5	+ 8.3
(2)	6.0454	5.7212	+27.9	+17.8	+15.6	+15.2	+14.6	+14.0	+13.5	+12.5
(3)	6.0178	5.6904	+18.4	+11.7	+10.2	+10.0	+ 9.7	+ 9.4	+ 9.2	+ 8.8
(4)	5.9830	5.6515	— 2.8	— 1.8	— 1.6	— 1.5	— 1.5	— 1.5	— 1.5	— 1.5
(5)	5.9541	5.6191	—22.7	—14.5	—12.7	—12.0	—11.8	—11.6	—11.4	—11.0
(6)	5.9391	5.6019	—29.8	—19.0	—16.7	—15.7	—15.3	—14.9	—14.5	—13.7
(7)	5.9316	5.5934	—20.7	—13.2	—11.6	—10.9	—10.5	—10.1	— 9.7	— 8.9
(8)	5.9364	5.5985	— 0.7	— 0.5	— 0.4	— 0.4	— 0.4	— 0.3	— 0.3	— 0.3
(9)	5.9512	5.6151	+19.3	+12.3	+10.9	+10.2	+ 9.8	+ 9.4	+ 9.0	+ 8.2
(10)	5.9737	5.6405	+29.1	+18.6	+16.4	+15.3	+14.9	+14.5	+14.1	+13.3
(11)	5.9959	5.6656	+23.4	+14.9	+13.1	+12.3	+12.1	+11.9	+11.7	+11.3
(12)	6.0124	5.6842	+ 4.5	+ 2.8	+ 2.5	+ 2.4	+ 2.4	+ 2.3	+ 2.3	+ 2.2
(13)	6.0251	5.6968	—17.0	—10.8	— 9.5	— 8.9	— 8.8	— 8.7	— 8.6	— 8.4
(14)	6.0341	5.7083	—28.1	—17.8	—15.7	—14.7	—14.3	—13.9	—13.6	—13.0
(15)	6.0468	5.7224	—21.0	—13.4	—11.8	—11.0	—10.6	—10.2	— 9.8	— 9.0
Σ		45.3439	— .5	— .3	— .2	+ .3			— .3	
Σ'		45.3441	0	— .1	0	+ .8			— .1	

g	Log. k_0	Log. k_1	Log. k_2	Log. k_3	Log. k_4	Log. k_5	Log. k_6	Log. k_7
(0)	8.465272	8.60289	8.38621	8.14674	7.89481	7.6341	7.3679	7.0975
(1)	8.466247	8.60407	8.38777	8.14874	7.89694	7.6369	7.3712	7.1013
(2)	8.462637	8.59849	8.38030	8.13935	7.88563	7.6238	7.3561	7.0843
(3)	8.457236	8.59018	8.36903	8.12505	7.86829	7.6033	7.3326	7.0577
(4)	8.450550	8.58006	8.35509	8.10719	7.84645	7.5774	7.3026	7.0237
(5)	8.445362	8.57214	8.34391	8.09259	7.82837	7.5559	7.2776	6.9950
(6)	8.443224	8.56872	8.33868	8.08543	7.81922	7.5446	7.2645	6.9800
(7)	8.442508	8.56750	8.33651	8.08224	7.81495	7.5395	7.2581	6.9726
(8)	8.444020	8.56954	8.33902	8.08521	7.81840	7.5433	7.2623	6.9771
(9)	8.444679	8.57452	8.34564	8.09354	7.82847	7.5551	7.2760	6.9925
(10)	8.453274	8.58206	8.35573	8.10632	7.84401	7.5734	7.2971	7.0165
(11)	8.458368	8.58906	8.36522	8.11851	7.85895	7.5912	7.3176	7.0400
(12)	8.461465	8.59345	8.37153	8.12680	7.86927	7.6036	7.3320	7.0564
(13)	8.461922	8.59532	8.37468	8.13126	7.87506	7.6105	7.3405	7.0660
(14)	8.461886	8.59651	8.37704	8.13471	7.87957	7.6163	7.3472	7.0739
(15)	8.462852	8.59905	8.38088	8.13992	7.88616	7.6242	7.3564	7.0845
Σ		68.69172	66.90360	64.93175	62.85706	60.7165	58.5297	56.3095
Σ'		68.69184	66.90364	64.93185	62.85719	60.7166	58.5300	56.3096

Values of Quantities in the Development of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$.

g	Log. k_8	Log. k_9	K_1	K_3	K_7	$(Q-g)-K_1$		$2(Q-g)-K_2$		$3(Q-g)-K_3$		
			′	′	′			°	′	°	′	″
(0)	6.8240	6.5478	−0.1	−0.1	−0.1	359	25.1	358	49.5	358	13	55.0
(1)	6.8280	6.5522	+4.4	+4.4	+4.4	0	28.5	1	24.6	2	14	57.0
(2)	6.8092	6.5317	+6.0	+6.0	+6.0	1	26.5	3	31.0	5	27	34.4
(3)	6.7795	6.4988	+3.9	+3.9	+3.9	2	15.2	4	55.5	7	30	33.9
(4)	6.7414	6.4566	−0.6	−0.6	−0.6	2	46.3	5	28.8	8	12	2.4
(5)	6.7093	6.4209	−4.7	−4.7	−4.7	2	47.3	5	3.8	7	26	37.6
(6)	6.6921	6.4016	−6.2	−6.2	−6.2	2	9.9	3	39.1	5	16	57.0
(7)	6.6837	6.3923	−4.3	−4.3	−4.3	0	55.7	1	23.2	1	56	39.0
(8)	6.6887	6.3976	−0.2	−0.2	−0.2	359	17.6	358	34.3	357	51	3.3
(9)	6.7058	6.4165	+4.0	+4.0	+4.0	357	36.2	355	38.7	353	35	39.5
(10)	6.7327	6.4463	+6.1	+6.1	+6.1	356	13.4	353	6.5	349	51	14.9
(11)	6.7589	6.4752	+5.0	+5.0	+5.0	355	26.8	351	25.5	347	17	29.4
(12)	6.7773	6.4958	+1.0	+1.0	+1.0	355	24.4	350	55.0	346	24	12.8
(13)	6.7883	6.5081	−3.5	−3.5	−3.5	356	1.7	351	40.2	347	23	40.2
(14)	6.7976	6.5187	−6.0	−6.0	−6.0	357	4.6	353	30.7	350	5	3.8
(15)	6.8093	6.5317	−4.5	−4.5	−4.5	358	15.9	356	3.1	353	56	22.5
Σ	54.0630	51.7961	.0	.0	.0	1793	47.8			1781	22	3.6
Σ'	54.0628	51.7957	+ .3	+ .3	+ .3	1433	47.3			1421	21	59.1

g	$4(Q-g)-K_4$		$5(Q-g)-K_5$		$6(Q-g)-K_6$		$7(Q-g)-K_7$		$8(Q-g)-K_8$		$9(Q-g)-K_9$	
	°	′	°	′	°	′	°	′	°	′		
(0)	357	38.5	357	3.0	356	27.5	355	52.1	355	16.7	354	41.2
(1)	3	3.9	3	53.2	4	42.5	5	31.8	6	21.1	7	10.5
(2)	7	22.4	9	17.4	11	12.4	13	7.3	15	2.2	16	57.1
(3)	10	4.4	12	38.3	15	12.2	17	46.0	20	19.8	22	53.6
(4)	10	55.5	13	39.0	16	22.5	19	6.0	21	49.5	24	33.0
(5)	9	50.6	12	15.0	14	39.4	17	3.9	19	28.4	21	52.8
(6)	6	55.9	8	35.6	10	15.3	11	54.9	13	34.5	15	14.2
(7)	2	30.9	3	5.5	3	40.1	4	14.7	4	49.3	5	23.9
(8)	357	8.0	356	24.9	355	41.7	354	58.6	354	15.5	353	32.4
(9)	351	31.8	349	27.7	347	23.6	345	19.5	343	15.4	341	11.3
(10)	346	34.9	343	17.8	340	0.7	336	43.7	333	26.7	330	9.6
(11)	343	8.5	338	58.9	334	49.3	330	39.7	326	30.1	322	20.5
(12)	341	53.2	337	22.1	332	51.1	328	20.0	323	48.9	319	17.9
(13)	343	7.7	338	52.3	334	36.9	330	21.5	326	6.1	321	50.7
(14)	346	40.5	342	16.6	339	52.7	336	28.8	333	4.9	329	41.1
(15)	351	50.4	349	44.9	347	39.4	345	33.8	343	28.2	341	22.7
Σ											1744	6.5
Σ'											1384	6.0

In the expansion of $\mu\left(\frac{a}{\Delta}\right)$.

g	$\overset{(c)}{A}_0$	$\overset{(c)}{A}_1$	$\overset{(s)}{A}_1$	$\overset{(c)}{A}_2$	$\overset{(s)}{A}_2$	$\overset{(c)}{A}_3$	$\overset{(s)}{A}_3$	$\overset{(c)}{A}_4$	$\overset{(s)}{A}_4$
	″	″	″	″	″	″	″	″	″
(0)	13.13109	6.9027	—.0701	+2.6281	—.0539	+1.10745	—.03418	+.4889	—.0201
(1)	13.13458	6.8933	+.0571	2.6294	+.0647	1.10917	+.04356	.4901	+.0262
(2)	13.11352	6.8033	+.1712	2.5849	+.1588	1.08348	+.10356	.4751	+.0615
(3)	13.08513	6.6912	+.2633	2.5254	+.2176	1.04890	+.13827	.4553	+.0809
(4)	13.05615	6.5922	+.3192	2.4646	+.2364	1.01333	+.14604	.4353	+.0840
(5)	13.03939	6.5457	+.3187	2.4259	+.2150	0.99004	+.12935	.4224	+.0733
(6)	13.04058	6.5584	+.2479	2.4172	+.1543	0.98367	+.09095	.4190	+.0509
(7)	13.04700	6.5880	+.1067	2.4198	+.0585	0.98375	+.03339	.4190	+.0184
(8)	13.05942	6.6190	—.0816	2.4317	—.0606	0.98937	—.03712	.4218	—.0211
(9)	13.07850	6.6377	—.2779	2.4464	—.1863	0.99667	—.11189	.4249	—.0633
(10)	13.10500	6.6498	—.4389	2.4645	—.2979	1.00593	—.18002	.4287	—.1023
(11)	13.12573	6.6578	—.5301	2.4816	—.3742	1.01487	—.22886	.4322	—.1310
(12)	13.13248	6.6727	—.5359	2.4991	—.3995	1.02497	—.24789	.4373	—.1431
(13)	13.12612	6.7090	—.4658	2.5224	—.3693	1.03984	—.23254	.4463	—.1354
(14)	13.11967	6.7727	—.3458	2.5559	—.2907	1.06142	—.18555	.4600	—.1090
(15)	13.12018	6.8478	—.2074	2.5954	—.1791	1.08668	—.11537	.4760	—.0683
Σ	104.75791	53.5708	—.7340	+20.0460	—.5531	8.26962	—.34421	+3.5661	—.1992
Σ'	104.75663	53.5705	—.7354	+20.0463	—.5531	8.26992	—.34409	+3.5662	—.1992

g	$\overset{(c)}{A}_5$	$\overset{(s)}{A}_5$	$\overset{(c)}{A}_6$	$\overset{(s)}{A}_6$	$\overset{(c)}{A}_7$	$\overset{(s)}{A}_7$	$\overset{(c)}{A}_8$	$\overset{(s)}{A}_8$	$\overset{(c)}{A}_9$	$\overset{(s)}{A}_9$
	″	″	″	″	″	″	″	″	″	″
(0)	+.2217	—.0114	+.1023	—.0063	+.0505	—.0036	+.0226	—.0019	+.0107	—.0010
(1)	.2223	+.0151	.1027	+.0085	.0498	+.0048	.0226	+.0025	.0108	+.0014
(2)	.2138	+.0350	.0978	+.0194	.0451	+.0105	.0211	+.0057	.0099	+.0030
(3)	.2028	+.0454	.0916	+.0249	.0401	+.0128	.0192	+.0071	.0089	+.0038
(4)	.1916	+.0465	.0856	+.0252	.0365	+.0126	.0176	+.0070	.0080	+.0037
(5)	.1848	+.0401	.0821	+.0215	.0356	+.0109	.0167	+.0059	.0076	+.0030
(6)	.1832	+.0277	.0815	+.0147	.0368	+.0078	.0166	+.0040	.0076	+.0021
(7)	.1833	+.0099	.0816	+.0052	.0384	+.0028	.0168	+.0014	.0077	+.0007
(8)	.1847	—.0116	.0823	—.0062	.0394	—.0035	.0169	—.0017	.0078	—.0009
(9)	.1860	—.0346	.0826	—.0185	.0388	—.0102	.0168	—.0051	.0077	—.0026
(10)	.1870	—.0561	.0827	—.0301	.0372	—.0160	.0166	—.0083	.0075	—.0043
(11)	.1880	—.0722	.0827	—.0389	.0354	—.0199	.0163	—.0108	.0072	—.0056
(12)	.1904	—.0793	.0837	—.0429	.0350	—.0216	.0163	—.0120	.0072	—.0062
(13)	.1956	—.0756	.0867	—.0411	.0369	—.0210	.0173	—.0116	.0077	—.0060
(14)	.2041	—.0613	.0918	—.0336	.0414	—.0180	.0190	—.0096	.0087	—.0051
(15)	.2140	—.0387	.0978	—.0214	.0468	—.0120	.0210	—.0062	.0098	—.0033
Σ	+1.5765	—.1105	+.7077	—.0598	+.3219	—.0318	+.1467	—.0168	+.0674	—.0087
Σ'	+1.5768	—.1106	+.7078	—.0598	+.3218	—.0318	+.1467	—.0168	+.0674	—.0086

In the expansion of $\mu a^2 \left(\frac{a}{\Delta}\right)^3$.

g	$\overset{(c)}{A}_0$	$\overset{(c)}{A}_1$	$\overset{(s)}{A}_1$	$\overset{(c)}{A}_2$	$\overset{(s)}{A}_2$	$\overset{(c)}{A}_3$	$\overset{(s)}{A}_3$	$\overset{(c)}{A}_4$	$\overset{(s)}{A}_4$
	″	″	″	″	″	″	″	″	″
(0)	23.3520	+32.0569	—0.3301	+19.4613	—0.4009	+11.2092	—0.3464	+6.269	—0.258
(1)	23.4045	32.1423	+0.4199	19.5273	+0.5272	11.2569	+0.4603	6.300	+0.347
(2)	23.2107	31.7192	+1.0033	19.1618	+1.2486	10.9731	+1.0737	6.096	+0.802
(3)	22.9239	31.1043	+1.3503	18.6375	+1.6470	10.5748	+1.4097	5.813	+1.041
(4)	22.5737	30.3821	+1.4503	18.0367	+1.7240	10.1342	+1.4580	5.516	+1.063
(5)	22.3056	29.8387	+1.2952	17.5937	+1.5122	9.8190	+1.2644	5.310	+0.912
(6)	22.1960	29.6180	+0.9110	17.4156	+1.0505	9.6988	+0.8734	5.239	+0.626
(7)	22.1595	29.5473	+0.3342	17.3564	+0.3782	9.6618	+0.3118	5.219	+0.222
(8)	22.2368	29.6867	—0.3713	17.4552	—0.4367	9.7264	—0.3654	5.259	—0.204
(9)	22.4249	30.0100	—1.1187	17.6808	—1.3068	9.8617	—1.0915	5.331	—0.786
(10)	22.7157	30.5036	—1.8033	18.0224	—2.1155	10.0630	—1.7762	5.436	—1.285
(11)	22.9837	30.9679	—2.3042	18.3471	—2.7150	10.2558	—2.2962	5.536	—1.667
(12)	23.1482	31.2707	—2.4810	18.5835	—2.9616	10.4121	—2.5144	5.627	—1.839
(13)	23.1725	31.4193	—2.3026	18.7500	—2.7837	10.5580	—2.3763	5.739	—1.748
(14)	23.1706	31.5386	—1.8212	18.9291	—2.2155	10.7412	—1.9027	5.895	—1.409
(15)	23.2222	31.7564	—1.1097	19.1791	—1.3716	10.9764	—1.1843	6.091	— .882
Σ	182.6038	246.7758	—3.4423	147.0656	—4.1071	82.9580	—3.5000	+45.337	—2.564
Σ'	182.5968	246.7862	—3.4356	147.0719	—4.1125	82.9644	—3.4985	+45.339	—2.563

g	$\overset{(c)}{A}_5$	$\overset{(s)}{A}_5$	$\overset{(c)}{A}_6$	$\overset{(s)}{A}_6$	$\overset{(c)}{A}_7$	$\overset{(s)}{A}_7$	$\overset{(c)}{A}_8$	$\overset{(s)}{A}_8$	$\overset{(c)}{A}_9$	$\overset{(s)}{A}_9$
	″	″	″	″	″	″	″	″	″	″
(0)	+3.440	—0.177	+1.863	—.115	+1.000	—.072	+.532	—.044	+.282	—.027
(1)	3.458	+0.240	1.874	+.157	1.005	+.098	.535	+.060	.283	+.036
(2)	3.318	+0.550	1.781	+.356	.944	+.221	.497	+.134	.260	+.076
(3)	3.130	+0.706	1.660	+.453	.868	+.279	.450	+.167	.231	+.098
(4)	2.937	+0.713	1.540	+.453	.797	+.276	.409	+.164	.208	+.095
(5)	2.812	+0.606	1.467	+.381	.756	+.232	.377	+.133	.196	+.078
(6)	2.772	+0.413	1.448	+.260	.748	+.157	.383	+.092	.195	+.053
(7)	2.766	+0.146	1.446	+.091	.750	+.055	.386	+.032	.197	+.019
(8)	2.789	—0.175	1.459	—.110	.757	—.053	.389	—.039	.199	—.023
(9)	2.824	—0.522	1.474	—.329	.760	—.199	.389	—.117	.197	—.067
(10)	2.870	—0.855	1.491	—.540	.759	—.326	.385	—.192	.193	—.111
(11)	2.915	—1.115	1.505	—.705	.757	—.425	.379	—.251	.187	—.144
(12)	2.963	—1.235	1.528	—.783	.767	—.473	.382	—.280	.188	—.162
(13)	3.042	—1.179	1.582	—.753	.803	—.457	.404	—.272	.201	—.158
(14)	3.164	—0.957	1.670	—.615	.867	—.378	.446	—.227	.227	—.133
(15)	3.312	—0.604	1.775	—.391	.942	—.243	.495	—.147	.259	—.087
Σ	24.253	—1.723	12.780	—1.094	+6.639	—.648	+3.423	—.392	+1.752	—.232
Σ'	24.259	—1.722	12.783	—1.095	6.641	—.660	+3.415	—.395	1.751	—.225

The Quantities $\frac{1}{2}\overset{(c)}{C}_{i,\nu}$, $\frac{1}{2}\overset{(s)}{C}_{i,\nu}$, $\frac{1}{2}\overset{(c)}{S}_{i,\nu}$, $\frac{1}{2}\overset{(s)}{S}_{i,\nu}$, arranged for Quadrature in the Expansion of

$$\mu\left(\frac{a}{\Delta}\right).$$

		$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
		″	″	″	″	″	″	″
$\nu=0$	$\overset{(c)}{C}_{i,0}$	+½[209.51454]	+53.571	+20.046	+8.26978	+3.566	+1.576	+.707
	$\overset{(c)}{S}_{i,0}$		—.735	—.553	—.34414	—.199	—.110	—.060
$\nu=1$	$\overset{(c)}{C}_{i,1}$	+.25653	+.548	+.382	+.22949	+.129	+.071	+.038
	$\overset{(s)}{S}_{i,1}$		+1.706	+1.273	+.78997	+.456	+.253	+.138
	$\overset{(s)}{C}_{i,1}$	—.25027	—.122	—.046	—.01129	+.002	+.005	+.006
	$\overset{(c)}{S}_{i,1}$		+.022	+.017	+.00807	+.003	+.001	.000
$\nu=2$	$\overset{(c)}{C}_{i,2}$	+.00463	+.257	+.096	+.05847	+.038	+.024	+.013
	$\overset{(s)}{S}_{i,2}$		—.170	—.003	+.01835	+.017	+.007	+.004
	$\overset{(s)}{C}_{i,2}$	+.12279	+.128	+.080	+.04667	+.026	+.015	+.001
	$\overset{(c)}{S}_{i,2}$		+.065	+.048	+.03063	+.018	+.010	+.006
$\nu=3$	$\overset{(c)}{C}_{i,3}$	+.03070	+.020	+.007	+.00662	+.005	+.002	+.001
	$\overset{(s)}{S}_{i,3}$		—.003	+.002	+.00216	+.002	+.001	+.001
	$\overset{(s)}{C}_{i,3}$	+.05945	+.041	+.023	+.01319	+.006	+.003	+.002
	$\overset{(c)}{S}_{i,3}$		000	—.001	—.00217	—.002	—.001	—.001
$\nu=4$	$\overset{(c)}{C}_{i,4}$	+.00037	+.001		+.00030			
	$\overset{(s)}{S}_{i,4}$		000		+.00052			
	$\overset{(s)}{C}_{i,4}$	+.00055	000		+.00076			
	$\overset{(c)}{S}_{i,4}$		—.001		—.00103			

The Quantities $\frac{1}{2}\overset{(c)}{C}_{i,\nu}$, $\frac{1}{2}\overset{(s)}{C}_{i,\nu}$, $\frac{1}{2}\overset{(c)}{S}_{i,\nu}$, $\frac{1}{2}\overset{(s)}{S}_{i,\nu}$, arranged for Quadrature, in the Expansion of

$$\mu a^2\left(\frac{a}{\Delta}\right)^3$$

		$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
		″	″	″	″	″	″	″	″	″	″
$\nu=0$	$\overset{(c)}{C}_{i,0}$	$+\frac{1}{2}[364.6002]$	+246.7810	+147.068	+82.9613	+45.338	+24.256	+12.781	+6.640	+3.419	+1.751
	$\overset{(c)}{S}_{i,0}$		—3.4388	—4.110	—3.4992	—2.562	—1.722	—1.095	—.654	—.392	—.228
$\nu=1$	$\overset{(c)}{C}_{i,1}$	+4.3500	+4.6277	+3.873	+2.8862	+1.956	+1.253	+.771	+.461	+.270	+.154
	$\overset{(s)}{S}_{i,1}$		+7.8438	+9.373	+7.9505	+5.816	+3.910	+2.488	+1.514	+.898	+.521
	$\overset{(s)}{C}_{i,1}$	—1.8014	—1.1511	—.801	—.3643	—.106	+.017	+.062	+.078	+.058	+.049
	$\overset{(c)}{S}_{i,1}$		+.1015	+.104	+.0731	+.043	+.024	+.011	—.008	+.003	+.001
$\nu=2$	$\overset{(c)}{C}_{i,2}$	—.2566	+.0899	+.294	+.3888	+.384	+.327	+.252	+.193	+.134	+.086
	$\overset{(s)}{S}_{i,2}$		+.1010	+.296	+.3297	+.302	+.239	+.173	+.116	+.078	+.047
	$\overset{(s)}{C}_{i,2}$	+1.1803	+1.1209	+.883	+.6281	+.418	+.266	+.162	+.093	+.058	+.031
	$\overset{(c)}{S}_{i,2}$		+.3367	+.400	+.3459	+.255	+.170	+.106	+.065	+.034	—.018
$\nu=3$	$\overset{(c)}{C}_{i,3}$	+.1113	+.1140	+.099	+.0809	+.066	+.049	+.035	+.024	+.013	+.012
	$\overset{(s)}{S}_{i,3}$		—.0170	.000	+.0059	+.012	+.015	+.015	+.015	+.013	+.008
	$\overset{(s)}{C}_{i,3}$	+.5132	+.6602	+.317	+.2097	+.130	+.076	+.043	+.020	+.012	+.002
	$\overset{(c)}{S}_{i,3}$		—.0138	—.030	—.0344	—.032	—.027	—.020	—.005	—.010	—.005
$\nu=4$	$\overset{(c)}{C}_{i,4}$	+.0177	+.0085	+.003	+.0028	+.002	+.002	.000	+.001	.000	+.001
	$\overset{(s)}{S}_{i,4}$		+.0117	+.005	+.0061	+.005	+.006	+.004	+.004	+.001	+.001
	$\overset{(s)}{C}_{i,4}$	+.0182	+.0172	+.016	+.0134	+.010	+.006	+.005	+.003	—.002	+.001
	$\overset{(c)}{S}_{i,4}$		—.0109	—.022	—.0182	—.016	—.012	—.008	+.002	—.003	—.001

The quantities $\overset{(c)}{C}_{i,\nu}$, $\overset{(s)}{C}_{i,\nu}$, etc., of the preceding tables have been divided by 2 to save division after quadrature. To check the values of these coefficients we will take the point corresponding to $g = 22^\circ.5$, using the equation

$$\overset{(c)}{A}_1, \text{ or } \overset{(s)}{A}_1 = \tfrac{1}{2}C_0 + C_1 \cos g + C_2 \cos 2g + \text{etc.}$$
$$+ S_1 \sin g + S_2 \sin 2g + \text{etc.},$$

noting that the tables give one-half of the values of these quantities.

Thus we have

	$i = 1$	$i = 2$		$i = 1$	$i = 2$
$\tfrac{1}{2}\overset{(c)}{C}_{1,0}$	$= +53''.571$	$+20''.046$	$\tfrac{1}{2}\overset{(c)}{S}_{1,0}$	$= -\ 0''.735$	$-\ 0''.553$
$\overset{(c)}{C}_{1,1}$	$= +\ 1.013$	$+\ .707$	$\overset{(s)}{S}_{1,1}$	$= +\ 1.306$	$+\ .974$
$\overset{(s)}{C}_{1,1}$	$= -\ .094$	$-\ .032$	$\overset{(c)}{S}_{1,1}$	$= +\ .040$	$+\ .031$
$\overset{(c)}{C}_{1,2}$	$= +\ .363$	$+\ .135$	$\overset{(s)}{S}_{1,2}$	$= -\ .240$	$-\ .004$
$\overset{(s)}{C}_{1,2}$	$= +\ .181$	$+\ .114$	$\overset{(c)}{S}_{1,2}$	$= +\ .092$	$+\ .070$
$\overset{(c)}{C}_{1,3}$	$= +\ .015$	$+\ .005$	$\overset{(s)}{S}_{1,3}$	$= -\ .005$	$+\ .004$
$\overset{(s)}{C}_{1,3}$	$= +\ .077$	$+\ .043$	$\overset{(c)}{S}_{1,3}$	$= 0$	$-\ .001$
$\overset{(c)}{C}_{1,4}$	$= 0$	..	$\overset{(s)}{S}_{1,4}$	$= 0$	..
$\overset{(s)}{C}_{1,4}$	$= 0$	..	$\overset{(c)}{S}_{1,4}$	$= 0$	..
Σ	$= +55''.126$	$+21''.018$	Σ	$= +\ 0''.458$	$+\ 0''.521$
$\tfrac{1}{8}\Sigma$	$= +\ 6.891$	$+\ 2.627$	$\tfrac{1}{8}\Sigma$	$= +\ 0.057$	$+\ 0.065$
$\overset{(c)}{A}_1$	$= +\ 6.893$	$+\ 2.629$	$\overset{(s)}{A}_1$	$= +\ 0.057$	$+\ 0.065$

In this way we check the values of these quantities for all values of i, in case of both $\mu\left(\frac{a}{\Delta}\right)$, and $\mu\alpha^2\left(\frac{a}{\Delta}\right)$.

Applying to the coefficients of the two preceding tables the formula

$$\left(\frac{a}{\Delta}\right)^n = \tfrac{1}{2}\Sigma\Sigma(\overset{(c)}{C}_{i,\nu} \mp \overset{(s)}{S}_{i,\nu}) \cos\left[(i\mp\nu)g - iE'\right] \mp \tfrac{1}{2}\Sigma\Sigma(\overset{(s)}{C}_{i,\nu} \pm \overset{(c)}{S}_{i,\nu}) \sin\left[(i\mp\nu)g - iE'\right]$$

noting that $\frac{1}{2}$ has been applied, we have the values of $\mu\left(\frac{a}{\Delta}\right)$, $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$ that follow:

	$\mu\left(\frac{a}{\Delta}\right)$		$\mu a^2\left(\frac{a}{\Delta}\right)^3$	
g E'	cos	sin	cos	sin
	$''$	$''$	$''$	$''$
0 0	$+\frac{1}{2}$[209.51455]		$+\frac{1}{2}$[364.6002]	
1 — 0	+0.25653	—0.25027	+4.3500	—1.8014
2 — 0	+0.00463	+0.12279	—0.2566	+1.1803
3 — 0	+0.03070	+0.05945	+0.1113	+0.5132
4 — 0	+0.00037	+0.00055	+0.0177	+0.0182
— 2 — 1	+0.023	—0.041	+0.1310	—0.6464
— 1 — 1	+0.427	—0.193	—0.0112	—1.4577
0 — 1	—1.158	+0.101	—3.2161	+1.0496
1 — 1	+53.571	+0.735	+246.7810	+3.4388
2 — 1	+2.254	—0.144	+12.4716	—1.2526
3 — 1	+0.087	+0.063	+0.1909	+0.7842
4 — 1	+0.016	+0.041	+0.0970	+0.6740
— 1 — 2			+0.099	—0.287
0 — 2	+0.098	—0.129	—0.001	—1.283
1 — 2	—0.891	+0.029	—5.500	+0.697
2 — 2	+20.046	+0.553	+147.068	+4.110
3 — 2	+1.656	—0.063	+13.246	—0.905
4 — 2	+0.093	+0.032	+0.590	+0.483
0 — 3	+0.00446	—0.01101	+.0750	—0.1753
1 — 3	+0.04011	—0.07730	+.0591	—0.9741
2 — 3	—0.56048	+0.00322	—5.0643	+0.2912
3 — 3	+8.26978	+0.34414	+82.9613	+3.4992
4 — 3	+1.01947	—0.01936	+10.8367	—0.4375
5 — 3	+0.07682	+0.01603	+0.7185	+0.2822
6 — 3	+0.00879	+0.01536	+0.0868	+0.2441
1 — 4	+0.003	—0.004	+0.053	—0.098
2 — 4	+0.020	—0.044	+0.082	—0.674
3 — 4	—0.326	—0.005	—3.859	+0.062
4 — 4	+3.566	+0.199	+45.338	+2.562
5 — 4	+0.585	—0.001	+7.772	—0.149
6 — 4	+0.055	+0.008	+0.687	+0.163
7 — 4			+0.078	+0.162
2 — 5	+0.005	+0.045	+0.033	—0.049
3 — 5	+0.016	—0.025	+0.088	—0.095
4 — 5	—0.182	—0.007	—2.657	—0.041
5 — 5	+1.576	+0.110	+24.256	+1.722
6 — 5	+0.325	+0.004	+5.163	—0.006
7 — 5	+0.031	+0.004	+0.567	+0.436
4 — 6	+0.009	—0.008	+0.079	—0.269
5 — 6	—0.100	—0.006	—1.717	—0.073
6 — 6	+0.707	+0.060	+12.781	+1.095
7 — 6	+0.176	+0.005	+3.260	+0.050
8 — 6	+0.018	—0.005	+0.426	+0.057

We have next to transform the expressions for $\mu\left(\frac{a}{\Delta}\right)$ and $\mu a^2\left(\frac{a}{\Delta}\right)^3$ just given into others in which both the angles involved are mean anomalies.

From

$$r_m = \frac{m}{h'\frac{e'}{2}},$$

beginning with $m = 5$, we find the values of r_5 for values of e' from $\frac{e'}{2}$ to e'^4.

Then we find

$$p_5 = \frac{1}{r_5}.$$

Putting $m = 4$, we find the values of r_4 as in the case of r_5. Then we get p_4 from

$$p_4 = \frac{1}{r_4 - p_5}.$$

We proceed in this way until we finally have the values of p_1. Then we find $\overset{(0)}{J}_{h'\frac{e'}{2}}$ or $\left(\overset{(0)}{J}_{h'\frac{e'}{2}} - 1\right)$ from

$$\overset{(0)}{J}_{h'\frac{e'}{2}} = 1 - l^2 + \frac{l^4}{4} - \frac{l^6}{36} \pm \text{etc.},$$

where $l = h'\frac{e'}{2}$,

and $\overset{(m)}{J}_{h'\frac{e'}{2}}$ from

$$\overset{(m)}{J}_{h'\frac{e'}{2}} = \overset{(0)}{J}_{h'\frac{e'}{2}} \cdot p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5.$$

The details of the computation are as follows:

Computation of the J functions.

$l =$	$\frac{1}{2}e'$	e'	$\frac{3}{2}e'$	$2e'$	$\frac{5}{2}e'$	$3e'$	$\frac{7}{2}e'$	$4e'$
log. l	8.38251	8.68354	8.85963	8.98457	9.08148	9.16066	9.22761	9.28560
log. r_5	2.31646	2.01543	1.83934	1.71440	1.61749	1.53831	1.47136	1.41337
log. p_5	7.68354	7.98457	8.16066	8.28560	8.38251	8.46169	8.52864	8.58663
log. r_4	2.21955	1.91852	1.74243	1.61749	1.52058	1.44140	1.37445	1.31646
log. r_4 — log. p_5	4.53601	3.93395	3.58177	3.33189	3.13807	2.97971	2.84581	2.72983
Zech	— 1	— 5	— 12	— 20	— 31	— 45	— 62	— 81
	2.21954	1.91847	1.74231	1.61729	1.52027	1.44095	1.37383	1.31585
log. p_4	7.78046	8.08153	8.25769	8.38271	8.47973	8.55905	8.62617	8.68415
log. r_3	2.09461	1.79358	1.61749	1.49255	1.39564	1.31646	1.24951	1.19152
Diff.	4.31415	3.71205	3.35980	3.10984	2.91591	2.75741	2.62334	2.50737
Zech	— 2	— 9	— 19	— 34	— 52	— 76	— 103	— 135
	2.09459	1.79349	1.61730	1.49221	1.39512	1.31570	1.24848	1.19017
log. p_3	7.90541	8.20651	8.38270	8.50779	8.60488	8.68430	8.75152	8.80983
log. r_2	1.91852	1.61749	1.44140	1.31646	1.21955	1.14037	1.07342	1.01543
Diff.	4.01311	3.41098	3.05870	2.80867	2.61467	2.45607	2.32190	2.20560
Zech	— 4	— 17	— 38	— 67	— 105	— 152	— 206	— 269
	1.91848	1.61732	1.44102	1.31579	1.21850	1.13885	1.07136	1.01274
log. p_2	8.08152	8.38268	8.55898	8.68421	8.78150	8.86115	8.92864	8.98726
log. r_1	1.61749	1.31646	1.14037	1.01543	0.91852	0.83934	0.77239	0.71440
Diff.	3.53597	2.93378	2.58139	2.33122	2.13702	1.97819	1.84375	1.72714
Zech	— 13	— 51	— 114	— 202	— 315	— 454	— 618	— 807
	1.61736	1.31595	1.13923	1.01341	0.91537	0.83480	0.76621	0.70633
log. p_1	8.38264	8.68405	8.86077	8.98659	9.08463	9.16520	9.23379	9.29367
log. l^4	3.53004	4.73716	5.43852	5.93828	6.32592	6.64264	6.91044	7.14240
log. $\frac{l^4}{4}$	2.92798	4.13210	4.83646	5.33622	5.72386	6.04058	6.30838	6.54034
— log. l^2	6.76502n	7.36708n	7.71926n	7.96914n	8.16296n	8.32132n	8.45522n	8.57120n
Diff.	3.83704	3.23498	2.88280	2.63292	2.43910	2.28084	2.14684	2.03086
Zech	— 7	— 25	— 57	— 101	— 157	— 227	— 308	— 402
log. $\left(-l^2+\frac{l^4}{4}\right)$	6.76495n	7.36693n	7.71869n	7.96813n	8.16139n	8.31905n	8.45214n	8.56718n
	3.23505	2.63307	2.28131	2.03187	1.83861	1.68095	1.54786	1.43282
Zech	— 26	— 101	— 227	— 401	— 625	— 896	— 1213	— 1575
log. $J^{(0)}$	9.99974	9.99899	9.99773	9.99599	9.99375	9.99104	9.98787	9.98425
log. p_1	8.38264	8.68405	8.86077	8.98659	9.08463	9.16520	9.23379	9.29367
log. $J^{(1)}$	8.38238	8.68304	8.85850	8.98258	9.07838	9.15624	9.22166	9.27792
log. p_2	8.08152	8.38268	8.55898	8.68421	8.78150	8.86115	8.92864	8.98726
log. $J^{(2)}$	6.46390	7.06572	7.41748	7.66679	7.85988	8.01739	8.15030	8.26518
log. p_3	7.90541	8.20651	8.38270	8.50779	8.60488	8.68430	8.75152	8.80983
log. $J^{(3)}$	4.36931	5.27223	5.80018	6.17458	6.46476	6.70169	6.90182	7.07501
log. p_4	7.78046	8.08153	8.25769	8.38271	8.47973	8.55905	8.62617	8.68415
log. $J^{(4)}$	2.14977	3.35376	4.05787	4.55729	4.94449	5.26074	5.52799	5.75916

Noting that $\log.(J^{(0)}-1)=\log.\left(-l^2+\frac{l^4}{4}\right)$, $\lambda'=\frac{e'}{2}$, and $l=h'\lambda'$, we form the following tables:

h'	$\text{Log.}\frac{1}{h'}\left(J^{(0)}_{h'\lambda'}-1\right)$	$\text{Log.}\frac{1}{h'}J^{(1)}_{h'\lambda'}$	$\text{Log.}\frac{1}{h'}J^{(2)}_{h'\lambda'}$	$\text{Log.}\frac{1}{h'}J^{(3)}_{h'\lambda'}$	$\text{Log.}\frac{1}{h'}J^{(4)}_{h'\lambda'}$
1	6.7649*n*	8.38238	6.4639	4.3693	2.1498
2	7.0658*n*	8.38201	6.7647	4.9712	3.0527
3	7.2415*n*	8.38138	6.9404	5.3231	3.5807
4	7.3661*n*	8.38052	7.0647	5.5725	3.9551
5	7.4624*n*	8.37941	7.1610	5.7658	4.2456
6	7.5409*n*	8.37809	7.2392	5.9235	4.4826
7	7.6070*n*	8.37656	7.3052	6.0567	4.6828
8	7.6641*n*	8.37483	7.3621	6.1719	4.8562

Value of $\frac{i'}{h'}J^{(h'-i')}_{h'\lambda'}$

i'	$h'=-2$	$h'=-1$	$h'=+1$	$h'=2$	$h'=3$	$h'=4$	$h'=5$	$h'=6$	$h'=7$	$h'=8$
1	4.9712*n*	6.4639*n*	6.76495*n*	8.38201	6.9404	5.5725	4.2455	....	....	....
2	3.3537*n*	4.6703*n*	8.68341*n*	7.36693*n*	8.68241	7.3657	6.0668	4.7835	....	....
3			6.9410	8.85913*n*	7.71869*n*	8.85764	7.6381	6.4006	5.1598	....
4			4.9714*n*	7.36675	8.98344*n*	7.96813*n*	8.98147	7.8413	6.6588	5.4583
5				5.6702*n*	7.6393	9.07949*n*	8.1614*n*	9.07706	8.0042	6.8709
6					6.1012*n*	7.8432	9.15756*n*	8.3190*n*	9.15471	8.1402
7	For h′ =0,					6.4176*n*	8.0061	9.22320*n*	8.4521*n*	9.21993
8	we have						6.6689*n*	8.1423	9.27965*n*	8.5672*n*
9		8.38251*n*						6.8777*n*	8.2594	9.32905*n*

In computing the values of the J functions, the lines headed Zech show that addition or subtraction tables have been used. For convenience, $(J^{(0)}-1)$ is employed instead of $J^{(0)}$, its values being found in the line headed $\log.\left(-l^2+\frac{l^4}{4}\right)$.

From the expression

$$((i, h')) = \Sigma \frac{i'}{h'} J_{h'\lambda'}^{(h'-i')} (i, i'),$$

h' being the multiple of g', and being constant, and i' being variable, we have

$$((i, h')) = \frac{1}{h'} J_{h'\lambda'}^{(h'-1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - E') + \frac{2}{h'} J_{h'\lambda'}^{(h'-2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - 2E') + \text{etc.}$$

$$- \frac{1}{h'} J_{h'\lambda'}^{(h'+1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') - \frac{2}{h'} J_{h'\lambda'}^{(h'+2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + 2E') - \text{etc.}$$

Now for $h' = +1$, we have, if we write the angle in place of the coefficient,

$$((ig - g')) = \tfrac{1}{1} J_{\lambda'}^{(0)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - E') + \tfrac{2}{1} J_{\lambda'}^{(-1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - 2E') + \text{etc.}$$

$$- \tfrac{1}{1} J_{\lambda'}^{(2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') - \tfrac{2}{1} J_{\lambda'}^{(3)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + 2E') - \text{etc.};$$

and for $h' = -1$, we have

$$((ig + g')) = - \tfrac{1}{1} J_{-\lambda'}^{(-2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - E') - \tfrac{2}{1} J_{-\lambda'}^{(-3)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - 2E') - \text{etc.}$$

$$+ \tfrac{1}{1} J_{-\lambda'}^{(0)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') + \tfrac{2}{1} J_{-\lambda'}^{(1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + 2E') + \text{etc.}$$

Since

$$J_{h'}^{(-m)} = (-1)^m J_{h'}^{(m)}, \qquad J_{-h'}^{(m)} = (-1)^m J_{h'}^{(m)}, \qquad J_{-h'}^{(-m)} = J_{h'}^{(m)},$$

the last two expressions give

$$((ig - g')) = J_{\lambda'}^{(0)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - E') - 2J_{\lambda'}^{(1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - 2E') \pm \text{etc.}$$

$$- J_{\lambda'}^{(2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') - 2J_{\lambda'}^{(3)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') - \text{etc.},$$

$$((ig + g')) = - J_{\lambda'}^{(2)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - E') - 2J_{\lambda'}^{(3)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig - 2E') - \text{etc.}$$

$$+ J_{\lambda'}^{(0)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + E') - 2J_{\lambda'}^{(1)} \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (ig + 2E') \pm \text{etc.}$$

And for the particular case of $i=1$, we have

$$\big((g-g')\big) = J_{\lambda'}^{(0)} {}^{\cos}_{\sin}(g-E') - 2J_{\lambda'}^{(1)} {}^{\cos}_{\sin}(g-2E') + 3J_{\lambda'}^{(2)} {}^{\cos}_{\sin}(g-3E') \mp \text{etc.}$$

$$- J_{\lambda'}^{(2)} {}^{\cos}_{\sin}(g+E') - 2J_{\lambda'}^{(3)} {}^{\cos}_{\sin}(g+2E') - 3J_{\lambda'}^{(4)} {}^{\cos}_{\sin}(g+3E') - \text{etc.}$$

$$\big((g+g')\big) = -J_{\lambda'}^{(2)} {}^{\cos}_{\sin}(g-E') - 2J_{\lambda'}^{(3)} {}^{\cos}_{\sin}(g-2E') - 3J_{\lambda'}^{(4)} {}^{\cos}_{\sin}(g-3E') - \text{etc.}$$

$$+ J_{\lambda'}^{(0)} {}^{\cos}_{\sin}(g+E') - 2J_{\lambda'}^{(1)} {}^{\cos}_{\sin}(g+2E') + 3J_{\lambda'}^{(2)} {}^{\cos}_{\sin}(g+3E') \mp \text{etc.}$$

Instead of $J_{\lambda'}^{(0)}$, we use $(J_{\lambda'}^{(0)} - 1)$, as has been noted.

If we put $h' = +2$, we have

$$\big((ig-2g')\big) = \tfrac{1}{2} J_{2\lambda'}^{(1)} {}^{\cos}_{\sin}(ig-E') + \tfrac{2}{2} J_{2\lambda'}^{(0)} {}^{\cos}_{\sin}(ig-2E') + \tfrac{3}{2} J_{2\lambda'}^{(-1)} {}^{\cos}_{\sin}(ig-3E') + \text{etc.}$$

$$- \tfrac{1}{2} J_{2\lambda'}^{(3)} {}^{\cos}_{\sin}(ig+E') - \tfrac{2}{2} J_{2\lambda'}^{(4)} {}^{\cos}_{\sin}(ig+2E') - \text{etc.}$$

In the table giving the values of $\frac{i'}{h'} J_{h'\lambda'}^{(h'-i')}$, we have, under $h'=2$, which applies to the equation just given,

for $i'=1$,	$\log.\ \tfrac{1}{2} J_{2\lambda'}^{(1)} = 8.38201$	$\log.\ (-\tfrac{1}{2} J_{2\lambda'}^{(3)}) = 4.9712n;$
for $i'=2$,	$\log.\ (\tfrac{2}{2} J_{2\lambda'}^{(0)} - 1) = 7.36693n$	$\log.\ (-\tfrac{2}{2} J_{2\lambda}^{(4)}) = 3.3537n;$
for $i'=3$,	$\log.\ (-\tfrac{3}{2} J_{2\lambda'}^{(1)}) = 8.85913n$	etc. = etc.
etc.,	etc. = etc.	

We find the values of $-\tfrac{1}{2} J_{2\lambda'}^{(3)}$, $-\tfrac{2}{2} J_{2\lambda'}^{(4)}$ in the table under $h' = -2$. We see that these are the forms of the function $\frac{i'}{h'} J_{h'\lambda}^{(h'-i')}$ when $h = -2$, and $i' = 1$ and $i' = 2$.

In the expansion of the coefficient of $(ig - h'g')$ indicated above by $\big((ig - h'g')\big)$, we have coefficients of angles of the form $(ig + i'E')$. These can readily be put into the form $(-ig - i'E')$, but the form employed is convenient in the transformation.

Arranging the functions $\mu\left(\frac{a}{\Delta}\right)$, $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$ in this form, we have

	Log. $\mu\left(\frac{a}{\Delta}\right)$		Log. $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$	
g E'	cos	sin	cos	sin
0 — 1	0.0637n	9.0043	0.5074n	0.0210
0 — 2	8.9912	9.1106n	7.0000n	0.1082n
0 — 3	7.6493	8.0418n	8.8751	9.2437n
1 + 1	9.6304	9.2856	8.0493	0.1637
1 — 1	1.72893	9.8663	2.3923	0.5364
1 — 2	9.9499n	8.4624	0.7404n	9.8432
1 — 3	8.6032	8.8882n	8.7716	9.9886n
1 — 4	7.4771	7.6021n	8.7243	8.9912n
2 + 1	8.3617	8.6128	9.1173	9.8105
2 — 1	0.3530	9.1584n	1.0959	0.0978n
2 — 2	1.30203	9.7427	2.1675	0.6138
2 — 3	9.7486n	7.5079	0.7045n	9.4642
2 — 4	8.3010	8.6435n	8.9138	9.8287n
2 — 5	6.6990	7.6532		
3 — 1	8.9395	8.7993	9.2808	9.8944
3 — 2	0.2191	8.7993n	1.1221	9.9566n
3 — 3	0.91750	9.5368	1.9189	0.5440
3 — 4	9.5132n	7.6990n	0.5865n	8.7924
3 — 5	8.2041	8.3979n	8.9445	8.9777n
4 — 1	8.2041	8.6128	8.9868	9.8287
4 — 2	8.9685	8.5051	9.7709	9.6839
4 — 3	0.0082	8.2869n	1.0348	9.6410n
4 — 4	0.5522	9.2989	1.6565	0.4085
4 — 5	9.2601n	7.8451n	0.4244n	8.6128n
4 — 6	7.9542	7.9093n	8.8976	9.4298n
5 — 3	8.8855	8.2049	9.8564	9.4506
5 — 4	9.7672	7.0000n	0.8905	0.1732n
5 — 5	0.1976	9.0414	1.3848	0.2360
5 — 6	9.0000n	7.7782n	0.2347n	8.8633n
6 — 3	7.9440	8.1864	8.9385	9.3876
6 — 4	8.7404	7.9031	9.8370	9.2122
6 — 5	9.5119	7.6021	0.7129	7.7782n
6 — 6	9.8494	8.7782	1.1066	0.0394
6 — 7			0.0224n	8.8451n
7 — 6			0.5132	8.6990
7 — 7			0.8222	9.8156
7 — 8			9.7973n	8.7924n

We will now give examples to illustrate the application of the tables for transforming from eccentric to mean anomaly, in case of the function $\mu\left(\frac{a}{\Delta}\right)$.

For the angle $3g - 3g'$.

	$\mu\left(\frac{a}{\Delta}\right)$		$\frac{i'}{h'}J^{(h'-i')}_{h'\lambda'}$				
g E'	cos	sin	$(h' = 3)$	Log. Product.		Product.	
						$''$	$''$
3 — 1	8.9395	8.7993	6.9404	5.8799	5.7397	+ .00008	+ .00005
3 — 2	0.2191	8.7993n	8.68241	8.9015	7.6817n	+ .07970	— .00303
3 — 3	0.91750	7.5368	7.71869n	8.6362n	5.2555n	— .04327	— .00180
3 — 4	9.5132n	7.6990n	8.98344n	8.4966	6.6824	+ .03139	+ .00048
3 — 5	8.2041	8.3979n	7.6393	5.8434	6.0372n	+ .00007	— .00011
						+8.26978	+0.34414
						+8.33775	+0.33973

For the angle $g - og'$.

			$(h' = 0)$			$''$	$''$
1 — 1	1.72893	9.8663	8.38251n	0.11144n	8.2488n	—1.29259	— .01773
1 + 1	9.6304	9.2856	8.38251n	8.0129n	7.6681n	— .01030	— .00466
						+0.25653	—0.25027
						—1.04636	—0.27266

For the angle $g + g'$.

			$(h' = -1)$			$''$	$''$
1 — 1	1.7289	9.8663	6.4639n	8.1928n	6.3302n	— .016	.000
						+0.427	+0.193
						+0.411	+0.193

For the angle $og - og'$.

						$''$	
$0-1$	$0.0637n$	....	$8.3825n$	8.4462	...	$+\quad .02794$	....
						$+104.75727$	....
						$+104.78521$	

For the angles represented by $(ig - g')$, there may be cases when there are sensible terms arising from $g + E'$, $g + 2E'$, etc.; if so, we use the column for $h' = -1$, and apply the proper numbers of this column to the coefficients of the angles named. Likewise in the case of $(ig + g')$, there may be terms arising from the product of the numbers in the column $h' = 1$ and the coefficients of the angles $g + E'$, etc. This will be made clear by an inspection of the two expressions

$$\begin{aligned}((ig - g')) = {} & \overset{(0)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig - E') - 2\overset{(1)}{J}_{\lambda} \tfrac{\cos}{\sin} (ig - 2E') \pm \text{etc.} \\ & - \overset{(2)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig + E') - 2\overset{(3)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig - 2E') - \text{etc.}, \\ ((ig + g')) = {} & - \overset{(2)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig - E') - 2\overset{(3)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig - 2E') - \text{etc.} \\ & + \overset{(0)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig + E') - 2\overset{(1)}{J}_{\lambda'} \tfrac{\cos}{\sin} (ig + 2E') \pm \text{etc.};\end{aligned}$$

where $((ig - g'))$, $((ig + g'))$ represent not the angles but their coefficients.

In retaining the form $(ig + i'E')$ instead of the form $(- ig - i'E')$ we can perform the operations indicated without any change of sign in case of the sine terms.

Making the transformations as indicated above, we obtain the following expressions for the functions $\mu\left(\frac{a}{\varDelta}\right)$, and $\mu\alpha^2\left(\frac{a}{\varDelta}\right)^3$:

	$\mu\left(\frac{a}{\Delta}\right)$		$\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$	
g g'	cos	sin	cos	sin
	″	″	″	″
0 — 0	+104.78521		+182.3777	
1 — 0	— 1.04636	—0.27266	— 1.6046	—1.9194
2 — 0	— 0.05031	+0.12527	— 0.5606	+1.1949
3 — 0	+ 0.02860	+0.05793	+ 0.1067	+0.4943
—2 — 1			— 0.1274	—0.6468
—1 — 1	+ 0.411	—0.193	— 0.0830	—1.4558
0 — 1	— 1.162	+0.107	— 3.2141	+1.1107
1 — 1	+ 53.583	+0.734	+246.9027	+3.4023
2 — 1	+ 1.286	—0.171	+ 5.3656	—1.4496
3 — 1	+ 0.014	+0.066	— 0.3758	+0.8304
0 — 2	+ 0.070	—0.127	— 0.085	—1.242
1 — 2	+ 0.399	+0.053	+ 0.456	+0.848
2 — 2	+ 20.093	+0.551	+147.392	+4.049
3 — 2	+ 1.056	—0.086	+ 7.214	—1.137
4 — 2	+ 0.027	+0.033	— 0.086	+0.537
0 — 3	+ 0.00815	—0.01707	+ 0.0718	—0.2352
1 — 3	+ 0.04342	—0.07447	+ 0.0041	—0.9231
2 — 3	+ 0.40733	+0.03392	+ 2.0442	+0.5514
3 — 3	+ 8.338	+0.340	+ 83.537	+3.432
4 — 3	+ 0.675	—0.036	+ 6.432	—0.659
5 — 3	+ 0.028	+0.010	+ 0.079	+0.449
2 — 4	+ 0.027	—0.043	+ 0.050	—0.637
3 — 4	+ 0.275	+0.023	+ 2.174	+2.592
4 — 4	+ 3.628	+0.197	+ 46.016	+2.512
5 — 4	+ 0.397	—0.013	+ 4.828	—0.323
6 — 4	+ 0.021	+0.008	+ 0.156	+0.188
3 — 5	+ 0.020	—0.023	+ 0.080	—0.074
4 — 5	+ 0.167	+0.012	+ 1.762	+0.241
5 — 5	+ 1.623	+0.109	+ 24.829	+1.565
6 — 5	+ 0.224	—0.004	+ 3.306	—0.148
4 — 6	+ 0.012	—0.008	+ 0.077	—0.250
5 — 6	+ 0.092	+0.007	— 4.535	+0.150
6 — 6	+ 0.731	+0.059	+ 13.312	+1.085

The transformation should be carefully checked by being done in duplicate, or better by putting the angle $ig = 0$, in all the divisions of the two functions, having thus only the angles $(0 - E')$, $(0 - 2E')$, $(0 - 3E')$, etc., etc.; also $(0 - g')$, $(0 - 2g')$, etc. Adding the coefficients in each division of the functions before and after transformation, and operating on the sums before transformation as on single members of the sums, the results should agree with the sums of the divisions of the transformations given above.

The transformations of these functions were checked by being done in duplicate, but we will give the check in case of another planet. We have for the logarithms of the sums before transformation, and for the sums after transformation the following:

g E'	cos	sin	g g'	cos ″	sin
0 — 1	1.85407	1.62090n	0 — 1	+ 70.548	— 40.188
0 — 2	1.25778	1.51473n	0 — 2	+ 19.809	— 32.318
0 — 3	9.7024n	1.26993n	0 — 3	+ 0.906	— 19.352
0 — 4	0.7101n	0.9147n	0 — 4	— 4.540	— 9.263
0 — 5	0.6632n	0.3899n	0 — 5	— 4.707	— 3.313
0 — 6	0.4387n	9.0934	0 — 6	— 3.059	— 0.330
0 — 7	0.1222n	9.8069	0 — 7	— 0.623	+ 0.739
0 — 8	9.5965n	9.8865	0 — 8	— 0.071	+ 0.615

For the angle (0 — 1),		(0 — 2),		0 — 3.	
″	″	″	″	″	″
— 0.041	+ 0.024	+ 1.722	— 1.007	+ .062	— .037
— 0.873	+ 1.578	— .042	+ .076	+ .871	— 1.574
.000	— 0 016	+ .037	+ 1.346	+ .003	+ .097
+ 71.462	— 41.774	— .012	— .019	+ .494	+ .791
+ 70.548	— 40.188	+ 18.104	— 32.714	— .020	— .011
+ 70.573	— 40.196	+ 19.809	— 32.318	— .504	— 18.618
		+ 19.811	— 32.319	+ 0.906	+ 19.352
				+ 0.902	— 19.355

The numbers in the last line of each case are the sums of the divisions after conversion when ig is put $= 0$.

To have close agreement it is necessary that all sensible terms in the expansion of $\mu\left(\frac{a}{\Delta}\right)$ and $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$ be retained. In the expressions for these functions given a large number of terms and some groups of terms have been omitted as they produce no terms in the final results of sufficient magnitude to be retained.

In transforming a series it will be convenient to have the values of the J functions on a separate slip of paper, so that by folding the slip vertically we can form the products at once without writing the separate factors.

The numerical expressions for $\mu\left(\frac{a}{\Delta}\right)$ and $\mu\alpha^2\left(\frac{a}{\Delta}\right)^3$ being known, we need next to have those designated by (H) and (I), which represent the action of the disturbing body on the Sun.

To find (H) we use two methods to serve as checks. We have first

$$
\begin{aligned}
(H) = {} & \tfrac{1}{2}[h\gamma_1\gamma_1' + h'\delta_1\delta_1'] \cos(g - g') && - \tfrac{1}{2}[l\delta_1\gamma_1' + l'\gamma_1\delta_1'] \sin(g - g') \\
& + \tfrac{1}{2}[h\gamma_1\gamma_1' - h'\delta_1'] \cos(-g - g') && - \tfrac{1}{2}[l\delta_1\gamma_1' - l'\gamma_1\delta_1'] \sin(-g - g') \\
& + \tfrac{1}{2}\, h\gamma_0\gamma_1' \cos(-g') && - \tfrac{1}{2} l'\gamma_0\delta_1' \sin(-g') \\
& + 2[h\gamma_1\gamma_2' + h'\delta_1\delta_2'] \cos(g - 2g') && - 2[l\delta_1\gamma_2' + l'\gamma_1\delta_2'] \sin(g - 2g') \\
& + 2[h\gamma_1\gamma_2' - h'\delta_1\delta_2'] \cos(-g - 2g') && - 2[l\delta_1\gamma_2' - l'\gamma_1\delta_2'] \sin(-g - 2g') \\
& + 2\, h\gamma_0\gamma_2' \cos(-2g') && - 2l'\gamma_0\delta_2' \sin(-2g') \\
& + \tfrac{9}{2}[h\gamma_1\gamma_3' + h'\delta_1\delta_2'] \cos(g - 3g') && - \tfrac{9}{2}[l\delta_1\gamma_3' + l'\gamma_1\delta_3'] \sin(g - 3g') \\
& + \text{etc.}
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1 &= J_\lambda^{(0)} - J_\lambda^{(2)} & \delta_1 &= J_\lambda^{(0)} + J_\lambda^{(2)} \\
\gamma_2 &= \tfrac{1}{2}[J_{2\lambda}^{(1)} - J_{2\lambda}^{(3)}] & \delta_2 &= \tfrac{1}{2}[J_{2\lambda}^{(1)} + J_{2\lambda}^{(3)}] \\
\gamma_3 &= \tfrac{1}{3}[J_{3\lambda}^{(2)} - J_{3\lambda}^{(4)}] & \delta_3 &= \tfrac{1}{3}[J_{3\lambda}^{(2)} + J_{3\lambda}^{(4)}],
\end{aligned}
$$

and similar expressions for γ_1', δ_1', γ_2', δ_2', etc.; noting that $\gamma_0 = -3e$.

The other expression for (H) is

$$
\begin{aligned}
(H) = {} & \tfrac{1}{2}[h\gamma_1' - h'\delta_1'] \cos(-E - g') + \tfrac{1}{2}[l\gamma_1' - l'\delta_1'] \sin(-E - g') \\
& + \tfrac{1}{2}[h\gamma_1' + h'\delta_1'] \cos(E - g) \quad - \tfrac{1}{2}[l\gamma_1' + l'\delta_1'] \sin(E - g') \\
& - eh\gamma_1' \quad \cos(-g') \quad + el'\delta_1' \quad \sin(-g') \\
& + 2[h\gamma_2' - h'\delta_2'] \cos(-E - 2g') + 2[l\gamma_2' - l'\delta_2'] \sin(-E - 2g') \\
& + 2[h\gamma_2' + h'\delta_2'] \cos(E - 2g') \quad - 2[l\gamma_2' + l'\delta_2'] \sin(E - 2g') \\
& - 4eh\gamma_2' \quad \cos(-2g') \quad + 4el'\delta_2' \quad \sin(-2g') \\
& + \text{etc.} \qquad\qquad + \text{etc.}
\end{aligned}
$$

In both expressions for (H) we have

$$
\begin{aligned}
h &= {}^{\mu}\ k \cos(\Pi - K) \\
h' &= \frac{\mu}{a^2} \cos\phi \cos\phi'\, k_1 \cos(\Pi - K_1) = \tfrac{1}{2}\mu \frac{v \cos V}{a^3} \\
l &= \frac{\mu}{a^2} \cos\phi\, k \sin(\Pi - K) = \tfrac{1}{2}\mu \frac{v \sin V}{a^3} \\
l' &= \frac{\mu}{a^2} \cos\phi'\, k_1 \sin(\Pi - K) = \tfrac{1}{2}\mu \frac{p \cos P}{a^3}
\end{aligned}
$$

where as before

$$\mu = \frac{m'}{1+m} \,.\, 206264.''8 \quad \text{and} \quad \alpha = \frac{a'}{a}.$$

In the second expression the eccentric angle of the disturbed body appears and we must transform the expression into one in which both angles are mean anomalies. With the eccentricity, e, of the disturbed body we compute the J functions just as we did in case of e' of the disturbing body.

We have in case of Althæa

	$\frac{1}{2}e$	e	$\frac{3}{2}e$	$2e$
Log. $(\overset{(0)}{J}-1)$ =	7.20740n	7.80894n	8.16025n	8.40890n
Log. $\overset{(0)}{J}$ =	9.99930	9.99719	9.99368	9.98872
Log. $\overset{(1)}{J}$ =	8.60344	8.90341	9.07774	9.20016
Log. $\overset{(2)}{J}$ =	6.90632	7.5077	7.8587	8.1068
Log. $\overset{(3)}{J}$ =	5.0329	5.9356	6.4630	6.8365
Log. $\overset{(4)}{J}$ =	3.0347	4.2384	4.9418	5.4403

From these values we may form a table of $\frac{i}{h}\overset{(h-i)}{J_{h\lambda}}$ as was done for the disturbing body. The values of these quantities can be checked by means of the tables found in ENGELMANN'S edition of BESSEL'S *Werke*, Band I, pp. 103–109.

Finding the numerical value of (H) first by the second expression, we get

E g'	cos	sin
	$''$	$''$
1 — 1	+48.154	+0.651
—1 — 1	+ 0.188	—0.102
0 — 1	— 3.884	—0.044
1 — 2	+ 4.644	+0.062
—1 — 2	+ 0.018	—0.010
0 — 2	— 0.374	—0.004
1 — 3	+ 0.37800	+0.00510
—1 — 3	+ 0.00141	—0.00081
0 — 3	— 0.03048	—0.00036

To transform we change from $(hE - i'g')$ into $(i'g' - hE)$. Making the transformation, writing also the values found from the first expression for the sake of comparison, and the value of (I) which will next be determined, we have

g g'	(H) cos	(H) sin	(H) cos	(H) sin	(I) sin	(I) cos
	$''$	$''$	$''$	$''$	$''$	$''$
0 — 1	— 5.826	—0.066	— 5.824	—0.066	+4.799	+2.043
0 — 2	— 0.560	—0.006	— 0.562	—0.006	+0.463	+0.197
0 — 3	— 0.04566	—0.00057	— 0.04575	. . .	+0.038	+0.016
—1 — 1	+ 0.149	—0.103	+ 0.180	—0.103		
1 — 1	+48.076	+0.650	+48.079	+0.650		
1 — 2	+ 4.637	+0.062	+ 4.605	+0.062		
1 — 3	+ 0.37740	+0.00502	+ 0.37738	+0.00510		
2 — 1	+ 1.927	+0.026	+ 1.927	+0.030		
2 — 2	+ 0.186	+0.002	+ 0.186	+0.002		
2 — 3	+ 0.011	0.000	+ 0.015	0.000		

To find the numerical value of (I) needed in case of the function $a^2\left(\frac{d\Omega}{dZ}\right)$, we have

$$\begin{aligned}(I) = \quad & b\delta'_1 \sin(-\ g') + \quad b'\gamma'_1 \cos(-\ g') \\ & + 4\, b\delta'_2 \sin(-2g') + 4\, b'\gamma'_2 \cos(-2g') \\ & + 9\, b\delta'_3 \sin(-3g') + 9\, b'\gamma'_3 \cos(-3g') \\ & + \text{etc.} \qquad\qquad + \text{etc.}\end{aligned}$$

where

$$b = -\frac{\mu}{a^2} \cos\phi' \sin I \cos \Pi', \qquad b' = \frac{\mu}{a^2} \sin I \sin \Pi'.$$

Having the values of $\mu\left(\frac{a}{\Delta}\right)$, $\mu a^2\left(\frac{a}{\Delta}\right)^3$, (H), and (I), we next find those of

$$a\Omega, \qquad ar\frac{d\Omega}{dr}, \qquad \text{and } a^2\frac{d\Omega}{dz},$$

from

$$a\Omega \quad = \mu\left(\frac{a}{\Delta}\right) - (H)$$

$$ar\frac{d\Omega}{dr} = \tfrac{1}{2}\mu a^2\left(\frac{a}{\Delta}\right)^3\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right] - \tfrac{1}{2}\mu\left(\frac{a}{\Delta}\right) - (H)$$

$$a^2\frac{d\Omega}{dz} = -\mu a^2\left(\frac{a}{\Delta}\right)^3\frac{\sin I}{a}\cdot\frac{r'}{a'}\sin(f' + \Pi') + (I)$$

where

$$\frac{r^2}{a^2} = 1 + \tfrac{3}{2}e^2 - \tfrac{4}{1}\overset{(1)}{J_\lambda}\cos g - \tfrac{4}{4}\overset{(2)}{J_{2\lambda}}\cos 2g - \tfrac{4}{9}\overset{(3)}{J_{3\lambda}}\cos 3g - \text{etc.}$$

$$-\frac{\sin I}{a}\frac{r'}{a'}\sin(f' + \Pi') = -[\overset{(0)}{J_{\lambda'}} + \overset{(2)}{J_{\lambda'}}]\,c_1\sin g' - \tfrac{1}{2}[\overset{(1)}{J_{2\lambda'}} + \overset{(3)}{J_{2\lambda'}}]\,c_1\sin 2g' - \text{etc.}$$

$$+ \tfrac{3}{2}e'c_2 - [\overset{(0)}{J_{\lambda'}} - \overset{(2)}{J_{\lambda'}}]\,c_2\cos g' - \tfrac{1}{2}[\overset{(1)}{J_{2\lambda'}} - \overset{(3)}{J_{2\lambda'}}]\,c_2\cos 2g - \text{etc.}$$

c_1 and c_2 being given by the equations

$$c_1 = \frac{\sin I}{a}\cos\phi'\cos\Pi'$$

$$c_2 = \frac{\sin I}{a}\sin\Pi'.$$

We find

$$\tfrac{1}{2}\left[\frac{r'^2}{a'^2} - \frac{1}{a^2}\frac{r^2}{a^2}\right] = [9.5769400] - 2[8.38238]\cos g' - 2[6.46366]\cos 2g' - \text{etc.}$$

$$+ 2[7.99450]\cos g + 2[6.29667]\cos 2g + \text{etc.}$$

$$-\frac{\sin I}{a}\frac{r'}{a'}\sin(f' + \Pi') = [7.18046] + 2[8.39074]\sin g' + 2[6.77809]\sin 2g'$$

$$- 2[8.01941]\cos g' - 2[6.40668]\cos 2g'$$

In multiplying two trigonometric series together, called by HANSEN mechanical multiplication,

let α_λ the coefficients of the angles λx in case of the sine,

β_μ those of the angles μx in case of the cosine,

γ_ν those of the angles νy in case of the sine,

and δ_ρ those of the angles ρy in case of the cosine.

The following cases then occur:

$$\alpha_\lambda \sin \lambda x \,.\, \delta_\rho \cos \rho y = \tfrac{1}{2}\,\alpha_\lambda\delta_\rho \sin(\lambda x + \rho y) + \tfrac{1}{2}\alpha_\lambda\delta_\rho \sin(\lambda x - \rho y)$$

$$\beta_\mu \cos \mu x \,.\, \gamma_\nu \sin \nu y = \tfrac{1}{2}\,\beta_\mu\gamma_\nu \sin(\mu x + \nu y) - \tfrac{1}{2}\beta_\mu\gamma_\nu \sin(\mu x - \nu y)$$

$$\beta_\mu \cos \mu x \,.\, \delta_\rho \cos \rho y = \tfrac{1}{2}\,\beta_\mu\delta_\rho \cos(\mu x + \rho y) + \tfrac{1}{2}\beta_\mu\delta_\rho \cos(\mu x - \rho y)$$

$$\alpha_\lambda \sin \lambda x \,.\, \gamma_\nu \sin \nu y = -\tfrac{1}{2}\,\alpha_\lambda\gamma_\nu \cos(\lambda x + \nu y) + \tfrac{1}{2}\alpha_\lambda\gamma_\nu \cos(\lambda x - \nu y).$$

In every term of the second members the factor $\frac{1}{2}$ occurs. Hence before multiplying we resolve the coefficients of one of the factors into two terms, one of which is 2. Performing the operations indicated, we have the values of $a\Omega$, $ar\frac{d\Omega}{dr}$, $a^2\frac{d\Omega}{dz}$ that follow:

	$a\Omega$		$ar\left(\frac{d\Omega}{dr}\right)$		$a^2\left(\frac{d\Omega}{dz}\right)$	
g g'	cos	sin	cos	sin	cos	sin
	″	″	″	″	″	″
0 — 0	+104.78521	....	+16.5202	....	+0.2828	....
1 — 0	— 1.04636	—.27266	— 2.4398	— .6940	—2.6311	+6.0177
2 — 0	— .05031	+.12527	— .3040	+ .3928	— .059	+ .239
3 — 0	+ .02860	+.05793	+ .0274	+ .1494	— .017	— .017
—1 — 1	+ .231	—.090	— .431	— .355	.000	— .129
0 — 1	+ 4.662	+.173	— 1.166	+ .481	—1.743	—4.157
1 — 1	+ 5.504	+.084	+18.839	+ .190	+ .318	+ .068
2 — 1	— .641	—.201	— 1.652	— .577	—1.596	+3.580
3 — 1	+ .014	+.066	— .240	+ .288	— .059	+ .232
0 — 2	+ .632	—.121	+ .497	— .414	— .020	— .149
1 — 2	— 4.206	—.009	— 9.136	+ .200	—2.474	—6.095
2 — 2	+ 19.907	+.549	+45.566	+1.270	+ .095	— .067
3 — 2	+ 1.056	—.086	+ 1.642	— .441	— .922	+2.011
4 — 2	+ .027	+.033	— .115	+ .180	— .064	+ .194
0 — 3	+ .05390	—.01764	+ .0718	— .0602	— .030	+ .017
1 — 3	— .33396	—.07957	— .4443	— .3306	— .045	— .166
2 — 3	+ .39221	+.03380	— 2.1788	+ .1339	—1.424	—3.658
3 — 3	+ 8.338	+.340	+27.227	+1.087	— .064	— .134
4 — 3	+ .675	—.036	+ 1.796	— .269	— .519	+1.099
5 — 3	+ .028	+.016	+ .043	+ .157	— .042	+ .123
2 — 4	+ .027	—.043	— .054	— .210	— .046	— .146
3 — 4	+ .275	+.023	— .880	+ .908	— .784	—2.078
4 — 4	+ 3.628	+.197	+15.430	+ .882	— .038	— .106
5 — 4	+ .397	—.013	+ .883	— .137	— .282	+ .586
6 — 4	+ .021	+.008	— .013	+ .063	— .031	+ .083
3 — 5	+ .020	—.023	— .034	— .078	+ .020	— .130
4 — 5	+ .167	+.012	— .281	+ .044	— .411	—1.150
5 — 5	+ 1.623	+.109	+ 8.605	+ .543	+ .024	— .227
6 — 5	+ .224	—.004	+ 1.061	+ .064	— .158	+ .311
4 — 6	+ 0.012	—.008	— 0.075	—0.095		
5 — 6	+ .092	+.007	— 2.225	+ .026		
6 — 6	+ .731	+.059	+ 4.559	+ .386		

Having $a\Omega$ we differentiate relative to g, and obtain $a\frac{d\Omega}{dg}$.

We then form the three products, $A\,.\,a\frac{d\Omega}{dg}$, $B\,.\,ar\left(\frac{d\Omega}{dr}\right)$, $C\,.\,a^2\left(\frac{d\Omega}{dz}\right)$. To this end we find A, B, C, from

$$\begin{aligned} A = & -3 + 2\,[2 + e^2] \cos(\gamma - g) \\ & + 2\,[\tfrac{e}{2} + \tfrac{e^3}{8}] \cos(\gamma - 2g) \\ & - 2\,[5\tfrac{e}{2} + \tfrac{25e^3}{16}] \cos\gamma \\ & + 2\tfrac{e^2}{4} \cos(\gamma - 3g) \\ & + 2\tfrac{e^3}{6} \cos(\gamma - 4g) \\ & + \text{etc.} \end{aligned}$$

$$\begin{aligned} B = & -2\,[1 - \tfrac{e^2}{2}] \sin(\gamma - g) \\ & - 2\,[\tfrac{e}{2} + \tfrac{e^3}{8}] \sin(\gamma - 2g) \\ & - 2\,[\tfrac{e}{2} + \tfrac{7}{16}e^3] \sin\gamma \\ & - 2\,\tfrac{3}{8}e^2 \sin(\gamma - 3g) \\ & - 2\tfrac{e^3}{3} \sin(\gamma - 4g) \\ & - \text{etc.} \end{aligned}$$

$$\begin{aligned} C = & \;\;\, 2\,[\tfrac{1}{2} - \tfrac{1}{4}e^2] \sin(\gamma - g) \\ & + 2\,[\tfrac{e}{4} - \tfrac{3}{16}e^3] \sin(\gamma - 2g) \\ & + 2\,[-\tfrac{3}{4}e + \tfrac{3}{32}e^3] \sin\gamma \\ & + 2\,\tfrac{3}{16}e^2 \quad \sin(\gamma - 3g) \\ & + 2\,\tfrac{1}{6}e^3 \quad \sin(\gamma - 4g) \\ & + \text{etc.} \end{aligned}$$

The numerical values of A, B, C in case of Althæa are

$$\begin{aligned} A = & -3 \\ & + 2\,[0.302429] \cos(\gamma - g) \\ & + 2\,[8.604489] \cos(\gamma - 2g) \\ & - 2\,[9.304508] \cos\gamma \\ & + 2\,[7.2076] \quad \cos(\gamma - 3g) \end{aligned}$$

$$\begin{aligned} B = & -2\,[0.001399] \sin(\gamma - g) \\ & - 2\,[8.604489] \sin(\gamma - 2g) \\ & - 2\,[8.606234] \sin\gamma \\ & - 2\,[7.3836] \quad \sin(\gamma - 3g) \end{aligned}$$

$$\begin{aligned} C = & + 2\,[9.697567] \sin(\gamma - g) \\ & + 2\,[8.30066] \quad \sin(\gamma - 2g) \\ & - 2\,[8.77953] \quad \sin\gamma \\ & + 2\,[7.08265] \quad \sin(\gamma - 3g) \end{aligned}$$

For the three products we then have

			$A\,.\,a\left(\frac{d\Omega}{dg}\right)$		$B\,.\,ar\left(\frac{d\Omega}{dr}\right)$		$C\,.\,a^2\left(\frac{d\Omega}{dz}\right)$	
γ	g	g'	sin	cos	sin	cos	sin	cos
			″	″	″	″	″	″
1	0	— 0	+ 2.1035	—0.5371	+ 1.1341	—0.6804	—1.3464	—3.0038
1	1	— 0	— .012	+ .565	+ .4021	+ .3723	+ .1287	+ .2411
—1	1	— 0	— .2530	+ .0439	—32.9502	+ .0549	— .3877	— .4802
1	2	— 0	— .192	+ .299	— .0153	+ .1657	— .0049	+ .0228
—1	2	— 0	+ 2.079	— .597	— 1.1310	+ .6821	+1.2995	+2.9772
—1	3	— 0	+ .261	+ .457	— .1263	— .3720	+ .083	+ .2404
1	—2	— 1	+ .462	+ .181	+ .432	— .348	— .076	+ .243
1	—1	— 1	— .266	— .015	+ .453	+ .461	—1.881	+4.454
1	0	— 1	—10.992	+ .153	—18.335	+ .187	+ .354	— .642
—1	0	— 1	+ .462	+ .181	— .477	+ .349	— .228	+ .572
1	1	— 1	+ 3.680	— .815	+ .929	— .559	— .815	—1.785
—1	1	— 1	+ 1.119	— .013	— .449	— .476	+1.906	—4.470
1	2	— 1	— .342	+ .477	+ .306	+ .276	+ .067	+ .098
—1	2	— 1	—11.301	+ .249	+18.336	— .188	— .178	— .359
—1	3	— 1	+ 2.360	— .843	— .929	+ .559	+ .785	+1.760
—1	4	— 1	— .033	+ .381	— .264	— .276		
1	—1	— 2	+ .232	.000	— .232		— .060	+ .194
1	0	— 2	+ 6.837	+ .026	+ 7.300	+ .235	—1.230	+3.029
—1	0	— 2	....	....			— .001	+ .009
1	1	— 2	—80.684	+2.195	—45.412	+1.264	+ .178	— .371
—1	1	— 2	— .848	+ .002	+ .132	+ .406	— .139	+ .290
1	2	— 2	+ 1.633	— .735	— 3.470	— .384	— .467	—1.010
—1	2	— 2	+16.433	— .240	— 7.317	— .235	+1.239	—3.036
1	3	— 2	+ .422	+ .316	+ .048	+ .168	+ .024	+ .023
—1	3	— 2	—79.078	+2.254	+45.412	—1.264	— .053	— .273
—1	4	— 2	— 7.937	— .500	— .213	+ .384	+ .454	+ .981
—1	5	— 2	— .408	+ .255	+ .198	— .163		
1	0	— 3	+ .5985	— .1553	+ .4644	— .3261	— .0482	+ .157
1	1	— 3	— 2.6517	+ .1927	+ 1.1042	+ .1641	— .7083	+1.8160
—1	1	— 3	— .0661	+ .0161	+ .0541	+ .0737	+ .0123	+ .0180
1	2	— 3	—50.140	+1.905	—27.2994	+1.0854	+ .043	— .174
—1	2	— 3	+ .828	— .1733	— .5308	+ .3287	— .062	+ .136
1	3	— 3	— .380	— .492	— 2.8964	— .2201	— .256	— .558
—1	3	— 3	+ 3.482	— .073	— 1.1112	— .1645	+ .707	—1.818
1	4	— 3	+ .263	+ .190	— .115	+ .147	+ .010	+ .005
—1	4	— 3	—49.676	+2.079	+27.299	—1.083	+ .029	— .206
—1	5	— 3	— 6.395	— .264	+ 3.899	+ .217	+ .257	+ .534

		$A.a\left(\frac{d\Omega}{dg}\right)$		$B.ar\left(\frac{d\Omega}{dr}\right)$		$C.a^2\left(\frac{d\Omega}{dz}\right)$	
γ	g g'	sin	cos	sin	cos	sin	cos
		″	″	″	″	″	″
1	1 — 4	— .165	— .170	....		— .038	+ .115
1	2 — 4	— 2.229	+ .187	+ .264	+ .939	— .389	+1.029
—1	2 — 4	+ .011	+ .017	....	....	+ .008	+ .014
1	3 — 4	—29.032	+1.564	—15.481	+ .915	+ .022	— .083
—1	3 — 4	+ .058	— .187	— .089	+ .175	— .024	+ .051
1	4 — 4	— 1.063	— .287	— 1.504	— .098	— .140	— .300
—1	4 — 4	+ 1.268	— .024	— .022	— .938	+ .390	—1.033
—1	5 — 4	—28.751	+1.597	+15.479	— .915	+ .033	— .129
—1	6 — 4	— 4.543	— .108	+ 1.506	+ .098		
1	2 — 5	— .160	— .136			+ .002	+ .088
1	3 — 5	— 1.654	+ .132	— .063	+ .063	— .206	+ .570
—1	3 — 5	+ .012	+ .014	— .001	— .003	+ .001	+ .008
1	4 — 5	—16.185	+1.082	— 8.661	+ .544	+ .034	+ .038
—1	4 — 5	+ .015	— .148	— .045	+ .076	— .035	+ .004
1	5 — 5	— 1.061	— .158	— 1.412	— .036	— .080	— .168
—1	5 — 5	+ .294	— .017	+ .062	— .063	+ .206	— .563
—1	6 — 5	—16.038	+1.100	+ 8.661	— .544		
1	3 — 6	— .121	— .063				
1	4 — 6	— 1.088	+ .086	+ 2.052	+ .038		
1	5 — 6	— 8.707	+ .703	— 4.516	+ .387		
—1	7 — 6	— 8.818	+ .711	+ 4.516	— .387		

Next from

$$\frac{dW}{ndt} = A.a\left(\frac{d\Omega}{dg}\right) + B.ar\left(\frac{d\Omega}{dr}\right)$$

we find the value of $\frac{dW}{ndt}$. Then we find W and $\frac{u}{\cos i}$ from

$$W = \int \frac{dW}{ndt}$$

$$\frac{u}{\cos i} = \int C.a^2\left(\frac{d\Omega}{dz}\right).$$

We first form a table giving the integrating factors. From log. $n' = 2.4758576$, log. $n = 2.9323542$, we have $\frac{n'}{n} = 0.34954524$.

i i'	$i + i'\frac{n'}{n}$	Log. $\left(i+i'\frac{n'}{n}\right)$	Log. $\left(\frac{1}{i+i\frac{n'}{n}}\right)$	i i'	$i + i'\frac{n'}{n}$	Log. $\left(i+i'\frac{n'}{n}\right)$	Log. $\left(\frac{1}{i+i'\frac{n'}{n}}\right)$
—2 — 1	—2.34954	0.37098n	9.62902n	3 — 3	+1.95136	0.29034	9.70966
—1 — 1	—1.34954	0.13018n	9.86982n	4 — 3	+2.95136	0.47002	9.52998
0 — 1	— .34954	9.54350n	0.45650n	5 — 3	+3.95136	0.5968	9.4032
1 — 1	+ .65045	9.813217	0.186783	1 — 4	— .398181	9.60008n	0.39992n
2 — 1	+1.65045	0.21760	9.78240	2 — 4	+ .601819	9.77946	0.22054
3 — 1	+2.65045	0.4233	9.5767	3 — 4	+1.601819	0.20461	9.79539
4 — 1	+3.65045	0.5624	9.4376	4 — 4	+2.601819	0.41528	9.58472
—1 — 2	—1.69909	0.23021n	9.76979n	5 — 4	+3.601819	0.5565	9.4435
0 — 2	— .69909	9.8446n	0.1554n	6 — 4	+4.601819	0.6630	9.3370
1 — 2	+ .30091	9.478423	0.521577	2 — 5	+ .252274	9.40187	0.59813
2 — 2	+1.30091	0.11425	9.88575	3 — 5	+1.252274	0.09770	9.90230
3 — 2	+2.30091	0.36190	9.63810	4 — 5	+2.252274	0.35263	9.64737
4 — 2	+3.30091	0.5186	9.4814	5 — 5	+3.252274	0.5122	9.4878
5 — 2	+4.30091	0.6336	9.3664	6 — 5	+4.252274	0.6286	9.3714
0 — 3	—1.04864	0.02062n	9.97938n	3 — 6	+ .902729	9.9556	0.0444
1 — 3	— .04863572	8.6869553n	1.3130447n	4 — 6	+1.902729	0.2794	9.7206
2 — 3	+ .95136	9.97835	0.02165	5 — 6	+2.902729	0.4628	9.5372

In regard to this table we may add that the form of the angles is $(ig + i'g') = \left(i + i'\frac{g'}{g}\right)g = \left(i + i'\frac{n'}{n}\right)nt$. The differential relative to the time is $\left(i + i'\frac{n'}{n}\right)ndt$.

The preceding table is applied by subtracting the logarithms of the column headed log. $\left(i + i'\frac{n'}{n}\right)$, or by adding the logarithms of the column headed log. $\left(\frac{1}{i + i\frac{n'}{n}}\right)$.

We will now give the values of $\frac{dW}{ndt}$, W, and $\frac{u}{\cos i}$, remarking that in the integrations the angle γ is constant; after the integrations it changes into g.

		$\frac{dW}{ndt}$		W		$\frac{u}{\cos i}$	
γ	g g'	sin	cos	cos	sin	cos	sin
		″	″	″	″	″	″
1	0 — 0	+ 3.2376	—1.2175	— 1.2175 nt	+ 3.2376 nt	—3.0038 nt	— 1.3464 nt
1	1 — 0	+ .3901	+ .9373	— .3901	+ .9373	— .1287	+ .2411
—1	1 — 0	+ 32.6972	+ .0988	— 32.6972	+ .0988	+ .3877	— .4802
1	2 — 0	— .2073	+ .4647	+ .1036	+ .2323	+ .0024	+ .0114
—1	2 — 0	+ .9480	+ .0851	— .4740	+ .0425	— .6497	+ 1.4886
—1	3 — 0	+ .1350	+ .0850	— .0450	+ .0283	— .028	+ .0801
1	— 2 — 1	+ .894	— .167	+ .383	+ .07	— .033	— .10
1	— 1 — 1	+ .187	+ .446	+ .115	— .330	—0.62	— 1.60
1	0 — 1	— 29.327	+ .340	— 83.900	— .973	+1.013	+ 1.84
—1	0 — 1	— .015	+ .530	— .045	— 1.516	— .652	— 1.64
1	1 — 1	+ 4.609	—1.374	— 7.087	— 2.112	+1.264	— 2.74
—1	1 — 1	+ .670	— .489	— 1.030	— .752	—1.370	— 3.21
1	2 — 1	— .036	+ .753	+ .022	+ .456	— .040	+ .06
—1	2 — 1	+ 7.035	+ .061	— 4.263	+ .038	+ .107	— .21
1	3 — 1	— .019	+ .254	+ .007	+ .096		
—1	3 — 1	+ 1.431	— .284	— .540	— .107	— .296	+ .670
—1	4 — 1	— .297	+ .105	+ .081	+ .029		
1	— 1 — 2					— .03	— .11
1	0 — 2	+ 14.145	+ .261	+ 20.207	— .373	—1.76	— 4.33
1	1 — 2	—126.276	+3.459	+419.660	+11.503	— .59	— 1.23
—1	1 — 2	— .716	+ .408	+ 2.380	+ 1.356	+ .46	+ .96
1	2 — 2	— 1.837	—1.119	+ 1.410	— .860	+ .36	— .78
—1	2 — 2	+ 9.116	— .475	— 7.008	— .365	— .95	— 2.34
1	3 — 2	+ .470	+ .484	— .204	+ .210	— .01	+ .01
—1	3 — 2	— 33.666	+ .990	+ 14.632	+ .430	+ .02	— .12
1	4 — 2	— .017	+ .125	+ .005	+ .038		
—1	4 — 2	— 8.150	— .116	+ 2.469	— .035	— .14	+ .30
—1	5 — 2	— .210	+ .092	+ .050	+ .021		
1	0 — 3	+ 1.0629	— .4814	+ 1.0136	+ .4591	— .05	— .15
1	1 — 3	— 1.5475	+ .3568	— 31.8180	— 7.335	—14.56	—37.33
—1	1 — 3	— .0120	+ .0898	— .2452	— 1.847	+ .25	— .37
1	2 — 3	— 77.4394	+2.9904	+ 81.400	+ 3.139	— .04	— .18
—1	2 — 3	+ .2972	+ .1554	— .3124	+ .1631	+ .06	+ .14
1	3 — 3	— 3.2764	— .7121	+ 1.679	— .365	+ .13	— .28
—1	3 — 3	+ 2.3706	— .2375	— 1.216	— .122	— .36	— .91
1	4 — 3	+ .148	+ .337	— .050	+ .115	.00	.00
—1	4 — 3	— 22.377	+ .996	+ 7.413	+ .338	— .01	— .07
—1	5 — 3	— 2.496	— .047	+ .627	— .012	— .06	+ .13
1	1 — 4	— .165		— .414		— .096	— .29
1	2 — 4	— 1.965	+1.126	+ 3.265	+ 1.871	+ .647	+ 1.71
1	3 — 4	— 44.513	+2.479	+ 27.790	+ 1.548	— .014	— .05
—1	3 — 4	— .031	— .012	+ .019	— .007	+ .015	+ .03
1	4 — 4	— 2.567	— .385	+ .986	— .148	+ .054	— .12
—1	4 — 4	+ 1.002	— .963	— .385	— .370	— .150	— .40
1	5 — 4	— .022	+ .057	+ .006	+ .016		
—1	5 — 4	— 13.272	+ .682	+ 3.686	+ .190	— .009	— .04
—1	6 — 4	— 3.037	— .010	+ .660	— .002		

	$\frac{dW}{ndt}$		W		$\frac{u}{\cos i}$	
γ g g'	sin	cos	cos	sin	cos	sin
	″	″	″	″	″	″
1 3 — 5	— 1.717	+ .195	+ 1.374	+ .156	+ .163	+ .46
—1 3 — 5	+ .011	+ .017	— .009	+ .014	— .001	+ .01
1 4 — 5	— 24.846	+1.626	+ 11.030	+ .722	— .015	+ .02
—1 4 — 5	— .030	— .072	+ .013	— .032	+ .016	.00
1 5 — 5	— 2.473	— .194	+ .760	— .060	+ .025	— .05
—1 5 — 5	+ .356	— .080	— .110	— .024	— .064	— .18
1 6 — 5	— .089	+ .160	+ .021	+ .038		
—1 6 — 5	— 7.377	+ .556	+ 1.735	+ .130		
—1 7 — 5	+ 1.413	+ .036	— .270	+ .007		
1 4 — 6	+ .964	+ .124	— .507	+ .07		
1 5 — 6	— 13.223	+1.090	+ 4.555	+ .38		
—1 5 — 6	— .167	+ .023	+ .057	+ .06		
1 6 — 6	— .946	— .002	+ .242	.00		
—1 6 — 6	— 2.098	— .040	+ .538	— .01		
—1 7 — 6	— 3.302	+ .324	+ .674	+ .09		

The part of W independent of γ arising from the factor, —3, in the value of A, has not yet been given. Its integral, or $\int -3a\left(\frac{d\Omega}{dg}\right)$, is the following:

$$\int - 3a\left(\frac{d\Omega}{dg}\right)$$

g g'	cos	sin	g g'	cos	sin
	″	″		″	″
1 — 0	+ 3.1392	+ .8181	4 — 3	— 2.74	+.14
2 — 0	+ .1509	— .3757	5 — 3	— .11	—.06
3 — 0	— .0858	— .1738			
			2 — 4	— .27	+.43
—1 — 1	— .51	+ .20	3 — 4	— 1.54	—.13
1 — 1	—25.39	— .39	4 — 4	—16.74	—.91
2 — 1	+ 2.33	+ .73	5 — 4	— 1.65	+.05
3 — 1	— .04	— .22	6 — 4	— .08	—.03
1 — 2	+41.934	+ .090	3 — 5	— .14	+.16
2 — 2	—91.80	—2.53	4 — 5	— .89	—.06
3 — 2	— 4.13	+ .34	5 — 5	— 7.49	—.50
4 — 2	— .10	— .12	6 — 5	— .95	+.02
1 — 3	—20.6020	—4.9099	4 — 6	— .07	+.05
2 — 3	— 2.473	— .210	5 — 6	— .48	—.04
3 — 3	—38.46	—1.57	6 — 6	— 3.35	—.27

Having the values of the coefficients of $(\pm\gamma + ig + i'g')$, both for W and $\frac{u}{\cos i}$, we have next to find those of $(\pm\nu\gamma + ig + i'g')$, and of $(0\gamma + ig + i'g)$ in the case of $\frac{u}{\cos i}$.

The expressions for this purpose are

$$\eta^{(2)} = \tfrac{1}{2}e - \tfrac{1}{8}e^3 - \tfrac{1}{384}e^5$$

$$\eta^{(3)} = \tfrac{3}{8}e^2 - \tfrac{15}{128}e^4$$

$$\eta^{(4)} = \tfrac{1}{3}e^3$$

$$\eta^{(0)} = -\left(\tfrac{3}{2}e + \tfrac{9}{16}e^3 \pm \text{etc.}\right)$$

For Althæa we find

$$\log.\ \eta^{(2)} = 8.60309 \qquad \log.\ \eta^{(3)} = 7.38368 \qquad \log.\ \eta^{(0)} = 9.08196n$$

We multiply the coefficients of $(\pm\gamma + ig + i'g')$ by $\eta^{(2)}$, and $\eta^{(3)}$, respectively, to find those of $(\pm 2\gamma + ig + i'g')$, $(\pm 3\gamma + ig + i'g')$.

In case of $(0\gamma + ig + i'g')$ in the expression for $\frac{u}{\cos i}$ we add the coefficients of $(+\gamma + ig + i'g')$ to those of $(-\gamma + ig + i'g)$ and multiply the sum by $\eta^{(0)}$.

We will give a few examples to show the formation of $\overline{W}$, and $-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$.

With these two we give at once also their integrals, which are $n\delta z$ and ν respectively.

$\overline{W}$ $\qquad\qquad -\frac{1}{2}\frac{d\,\overline{W}}{d\gamma}$

(0 — 0)

		$\overline{W}$ cos	$\overline{W}$ sin	$-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$ sin	$-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$ cos
		″	″	″	″
— 1	1 — 0	—32.6972	+.0988	+16.3486	+.0494
— 2	2 — 0	— .0190	+.0017	+ .0190	+.0017
		—32.7162			+.0511
		″			″
		—32.7162nt			+.0511nt

$\overline{W}$ $\qquad\qquad$ $-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$

(1 — 0)

		″	″	″	″
— 1	2 — 0	— .474	+ .042	+ .237	+ .021
0	1 — 0	+3.139	+ .818	—	—
2	— 1 — 0	—1.314	— .004	—1.314	+ .004
1	0 — 0	—1.2175nt	+3.2376nt	— .6087nt	—1.6188nt
		″ ″	″ ″	″ ″	″ ″
		+1.351 — 1.2175nt	+ .856 +3.2376nt	—1.077 —.6087nt	+ .025 —1.6188nt
		″ ″	″ ″	″ ″	″ ″
		+4.59 — 1.2175nt	—2.07 —3.2376nt	—0.54 +.6087nt	—0.58 —1.6188nt

(— 1 — 1)

		″	″	″	″
1	— 2 — 1	+ .383	+ .070	+.191	—.035
— 1	0 — 1	— .045	—1.516	+.022	—.758
— 2	1 — 1	— .041	— .030	+.041	—.030
0	— 1 — 1	— .513	+ .200	—	—
		—0.216	—1.246	+.254	—.823
		″	″	″	″
		+ .16	— .92	+.19	+.61

(1 — 1)

		″	″	″	″
— 2	3 — 1	— .022	— .004	+ .022	—.004
— 1	2 — 1	— 4.263	+ .038	+ 2.131	+.019
0	1 — 1	— 25.390	— .390	—	—
1	0 — 1	— 83.900	— .973	—41.950	+.486
		—113.574	—1.329	—39.798	+.501
		″	″	″	″
		—174.61	+2.04	+61.19	+0.77

In the integration we apply the proper factor to each term of $\overline{W}$, $-\frac{1}{2}\frac{d\overline{W}}{d\gamma}$, and obtain the values of $n\delta z$, ν, except in case of the terms $(ig + og')$.

Let us take the term $(g - og')$ or $(1 - 0)$, and let μ the integrating factor to be applied.

Let c, a, d, b, represent the cos, sin, nt cos, nt sin terms respectively.

Thus we have

c	d	a	b
$''$	$''$	$''$	$''$
$+1.351$	$-1.2175nt$	$+.856$	$+3.2376nt;$

and hence

μc	$\mu^2 b$	μd	$-\mu a$	$\mu^2 d$	$-\mu b$
$''$	$''$	$''$	$''$	$''$	$''$
$+1.351$	$+3.2376$	$-1.2175nt$	$-.856$	-1.2175	$-3.2376nt$

or, since μ is unity,

$''$	$''$	$''$	$''$
$+4.59$	$-1.2175nt$	-2.07	$-3.2376.$

In case of the term $(2 - 0)$, μ is $\frac{1}{2}$.

In the way indicated we derive the values of $n\delta z$, and ν. In the case of $\frac{u}{\cos i}$ we have the values at once without another integration as was necessary for $n\delta z$ and ν.

In the value of W given above the arbitrary constants of integration have not been applied.

We give these constants in the form

$$k_0 + k_1 \cos\gamma + k_2 \sin\gamma + \eta^{(2)} k_1 \cos 2\gamma + \eta^{(2)} k_2 \sin 2\gamma + \text{etc.}$$

Then in case of $-\frac{1}{2}\frac{dW}{d\gamma}$ we have

$$\tfrac{1}{2}k_1 \sin\gamma - \tfrac{1}{2}k_2 \cos\gamma + \eta^{(2)} k_1 \sin 2\gamma - \eta^{(2)} k_2 \cos 2\gamma \pm \text{etc.}$$

Having W from the integration of $\frac{dW}{ndt}$, we form $\overline{W}$ from the value of W and converting γ into g.

We thus have from the equation

$$\frac{dz}{dt} = 1 + \overline{W} + \frac{h_0}{h}\left(\frac{\nu}{1+\nu}\right)^2,$$

$$\begin{aligned} \frac{dz}{dt} = 1 + k_0 & \\ +(1''.351 + k_1)\cos g & \quad + (0''.856 + k_2)\sin g \\ -\ 1''.2175nt \cos g & \quad +\ 3''.2376nt \sin g \\ + (-''.284 + \eta^{(2)} k_1)\cos 2g & \quad + (0''.589 + \eta^{(2)} k_2)\sin 2g \\ -''.0488nt \cos 2g & \quad + ''.1298nt \sin 2g \\ \pm \text{etc.} & \quad \pm \text{etc.} \end{aligned}$$

In the second integration the constants of $n\delta z$ and ν are designated by C and N respectively, and the complete forms are

$$C + k_0 nt + \quad k_1 \sin g - k_2 \ \cos g + \tfrac{1}{2}\eta^{(2)} k_1 \sin 2g - \tfrac{1}{2}\eta^{(2)} k_2 \cos 2g \pm \text{etc.}$$

$$N \qquad - \tfrac{1}{2}k_1 \cos g - \tfrac{1}{2}k_2 \sin g - \tfrac{1}{2}\eta^{(2)} k_1 \cos 2g - \tfrac{1}{2}\eta^{(2)} k_2 \sin 2g - \text{etc.}$$

In case of the latitude the constants of integration have the form

$$l_0 + l_1 \sin g + l_2 \cos g.$$

We thus find

$$\begin{aligned}
nz = C &+ [1 + k_0 - 32''.7162]nt \\
&+ [4''.59 + k_1] \sin g && + [-2''.07 - k_2] \cos g \\
&- 1''.2175 nt \sin g && - 3''.2376 nt \cos g \\
&+ [-0''.11 + \tfrac{1}{2}\eta^{(2)} k_1] \sin 2g && + [-0''.31 - \tfrac{1}{2}\eta^{(2)} k_2] \cos 2g \\
&- 0''.0244 nt \sin 2g && - 0''.0649 nt \cos 2g \\
&\pm \text{etc.} && \pm \text{etc.}
\end{aligned}$$

$$\begin{aligned}
\nu = &+ 0''.0511 nt + N \\
&+ [-0''.54 - \tfrac{1}{2}k_1] \cos g && + [-0''.58 - \tfrac{1}{2}k_2] \sin g \\
&+ 0''.6087 nt \cos g && - 1''.6188 nt \sin g \\
&+ [0''.05 - \tfrac{1}{2}\eta^{(2)} k_1] \cos 2g && + [-''.24 - \tfrac{1}{2}\eta^{(2)} k_2] \sin 2g \\
&+ 0''.0244 nt \cos 2g && - 0''.0649 nt \sin 2g \\
&\pm \text{etc.} && \pm \text{etc.}
\end{aligned}$$

$$\begin{aligned}
\frac{u}{\cos i} = &\ l_0 + 0''.3616 && + 0''.3623 nt \\
&+ [1''.52 + l_1] \sin g && + [-0''.68 + l_2] \cos g \\
&- 1''.3464 nt \sin g && - 3''.0038 nt \cos g \\
&+ 0''.32 \sin 2g && - 0''.16 \cos 2g \\
&- 0''.0539 nt \sin 2g && - 0''.1204 nt \cos 2g \\
&\pm \text{etc.} && \pm \text{etc.}
\end{aligned}$$

The complete expressions for $n\delta z$, ν, $\frac{u}{\cos i}$ in tabular form are the following:

g — g'	$n\delta z$ sin	$n\delta z$ cos	ν cos	ν sin	$\frac{u}{\cos i}$ sin	$\frac{u}{\cos i}$ cos
0 — 0		$+k_0\, nt$	$+N$			$+l_0 + 0''.36$
		$-32''.7162nt$	$+\ 0''.0511nt$			$+\ ''.3623nt$
1 — 0	$-\ 4''.59 + k_1$	$-\ 2''.07 - k_2$	$-\ 0''.54 - \frac{1}{2}k_1$	$-\ ''.58 - \frac{1}{2}k_2$	$+\ 1''.52 + l_1$	$-\ ''.68 + l_2$
	$-\ 1''.2175nt$	$-\ 3''.2376nt$	$+\ 0''.6087nt$	$-\ 1''.6188nt$	$-\ 1''.3464nt$	$-\ 3''.0038nt$
2 — 0	$-\ 0''.11 + \frac{1}{2}\eta^{(2)}k_1$	$-\ ''.31 - \frac{1}{2}\eta^{(2)}k_2$	$+\ ''.05 - \frac{1}{2}\eta^{(2)}k_1$	$-\ ''.24 - \frac{1}{2}\eta^{(2)}k^2$	$+\ ''.32$	$-\ ''.16$
	$-\ 0''.0244nt$	$-\ ''.0649nt$	$+\ ''.0244nt$	$-\ ''.0649nt$	$-\ ''.0539nt$	$-\ ''.1204nt$
0 — 1	+ 3.10	− 3.09	+ 2.12	− 1.54	− 4.83	− 2.03
0 — 2	− 3.00	+ 1.92	− 1.30	− .95	+ 1.30	+ .61
0 — 3	+ 0.23	− 1.76	+ .12	+ .89	− .37	+ .25
1 — 1	−174.61	+ 2.04	+ 61.19	+ .77	+ 2.69	+ 1.26
2 — 2	+263.97	− 7.21	−156.21	− 4.24	− 1.15	− .57
3 — 3	+ 25.15	− 0.81	− 18.30	− .56	− 1.60	− .60
4 — 4	+ 5.71	− 0.35	− 4.68	− .29	+ .03	+ .02
5 — 5	+ 1.64	− 0.11	− 1.45	− .09		
6 — 6	+ .49	− .05	− .50	− .04		
1 — 2	+185.18	+ 2.10	− 43.27	+ .07	− 6.64	− 2.70
2 — 4	− 1.10	− .71	+ .36	− .01	− .47	− .17
1 — 3	+410.16	−87.44	+ 14.64	+ 3.15	+ 4.43	+ 1.73
2 — 1	− 5.25	+ .87	+ 4.02	+ .62	− 1.98	+ .99
2 — 3	− 37.24	+ 8.03	+ 16.07	+ 3.78	−38.24	−14.92
3 — 2	+ 6.77	+ .04	− 7.08	− .01	− .52	+ .20
3 — 4	+ .90	− .86	− 1.05	− .70	+ 1.31	+ .50
4 — 3	+ .92	+ .04	− .69	+ .05	− .24	+ .03
4 — 5	+ .17	− .03	− .33	− .04	+ .28	+ .10
5 — 4	+ .34	+ .01	− .38	.00		
−1 — 1	+ .16	− .92	+ .19	+ .61	− 1.62	− .63

The constants of integration are now to be so determined as to make the perturbations zero for the Epoch. The following equations fulfill this condition:

$$C + k_1 \sin g - k_2 \cos g + \tfrac{1}{2}\eta^{(2)} k_1 \sin 2g - \tfrac{1}{2}\eta^{(2)} k_2 \cos 2g \pm \text{etc.} + (n\delta z)_0 = g_0$$

$$k_0 + k_1 \cos g + k_2 \sin g + \eta^{(2)} k_1 \cos 2g + \eta^{(2)} k_2 \sin 2g + \text{etc.} + \frac{d}{ndt}(n\delta z)_0 = 0$$

$$N - \tfrac{1}{2}k_1 \cos g - \tfrac{1}{2}k_2 \sin g - \tfrac{1}{2}\eta^{(2)} k_1 \cos 2g - \tfrac{1}{2}\eta^{(2)} k_2 \sin 2g - \text{etc.} + (\nu)_0 = 0$$

$$+ \tfrac{1}{2}k_1 \sin g - \tfrac{1}{2}k_2 \cos g + \eta^{(2)} k_1 \sin 2g - \eta^{(2)} k_2 \cos 2g \pm \text{etc.} + \frac{d}{ndt}(\nu)_0 = 0$$

$$l_0 + l_1 \sin g + l_2 \cos g + \eta^{(2)} l_1 \sin 2g + \eta^{(2)} l_2 \cos 2g + \text{etc.} + \left(\frac{u}{\cos i}\right)_0 = 0$$

$$l_1 \cos g - l_2 \sin g + \eta^{(2)} l_1 \cos 2g - \eta^{(2)} l_2 \sin 2g \pm \text{etc.} + \frac{d}{ndt}\left(\frac{u}{\cos i}\right)_0 = 0$$

To find k_1 and k_2 we have

$$k_1 [\cos g - e + \eta^{(2)} \cos 2g + \eta^{(3)} \cos 3g + \text{etc.}] + k_2 [\sin g + \eta^{(2)} \sin 2g + \text{etc.}]$$
$$- 3Z_0 + 6(\nu)_0 + 4\frac{d}{ndt}(n\delta z)_0 = 0$$

$$k_1 [\sin g + 2\eta^{(2)} \sin 2g + 3\eta^{(3)} \sin 3g + \text{etc.}] - k_2 [\cos g + 2\eta^{(2)} \cos 2g + \text{etc.}]$$
$$+ 2\frac{d}{ndt}(\nu)_0 = 0$$

where

$$N = -\tfrac{2}{3}k_0 - \tfrac{e}{6}k_1 - \tfrac{1}{2}Z_0, \quad Z_0 = -32''.7162,$$

k_0 being found from

$$k_0 = ek_1 + 3Z_0 - 3\frac{d}{ndt}(n\delta z)_0 - 6(\nu)_0.$$

We have also

$$l_0 = -el_2.$$

The symbols $(n\delta z)_0$, $(\nu)_0$, etc., represent the values of $n\delta z$, ν, etc., at the Epoch.

To find the values of the angles $(ig + i'g)$ at the Epoch we have

$$g = 332^\circ\ 48'\ 53''.2$$
$$g' = 63\ \ \ 5\ \ 48\ .6$$

The long period inequality, 5 Saturn — 2 Jupiter, is included in the value of g'.

From these values of g and g' we find the various arguments of the perturbations. Then forming the sine and cosine for each argument, we multiply the sine and cosine coefficients of the perturbations by their appropriate sines and cosines.

In forming $\frac{d}{ndt}(n\delta z)$, etc., we can make use of the integrating factors, multiplying by the numbers in the column $\left(i + i'\frac{n'}{n}\right)$. Having their differential coefficients we proceed as in the case of $(n\delta z)$, etc.

We thus find

$$(n\delta z)_0 = +401''.7, \qquad (\nu)_0 = +180''.6, \qquad \left(\frac{u}{\cos i}\right) = -22''.6$$

$$\frac{d}{ndt}(n\delta z)_0 = -391''.6, \qquad \frac{d}{ndt}(\nu)_0 = +\ 70''.5, \qquad \frac{d}{ndt}\left(\frac{u}{\cos i}\right) = +41''.5.$$

And from these we have

$$k_1 = +412''.8, \qquad k_2 = -82''.9, \qquad k_0 = -26''.21, \qquad l_0 = 0''.0$$
$$l_1 = -\ 45''.2, \qquad l_2 = +\ 0''.4, \qquad N = +28''.3,$$
$$C = 332^\circ\ 44'\ 12''.6.$$

The new mean motion is found from $(1 - 32''.7162 - 26''.21)\,nt$, which gives $n = 855''.5196$. With this value of n we find the only change is in the coefficients of the argument $(1-3)$, having $+405''.29$ instead of $410''.16$, and $-86''.30$ instead of $-87''.44$.

The constant C now has the value

$$C = 332^\circ\ 44'\ 16''.3.$$

Introducing the values of the constants of integration into the expressions for nz, ν, and $\frac{u}{\cos i}$, we have

$$
\begin{array}{lllll}
nz & = 332^\circ\, 44'\, 16''.3 & & + 855''.5196\, t & \\
 & + 417''.4 & \sin g & + 80''.8 & \cos g \\
 & - \quad 1''.2175 & \sin g & - \quad 3''.2376 & \cos g \\
 & + \quad 16''.4 & \sin 2g & + \quad 3''.0 & \cos 2g \\
 & - \quad 0''.0244\, nt & \sin 2g & - \quad 0''.0649\, nt & \cos 2g \\
 & \pm \text{ etc.} & & \pm \text{ etc.} &
\end{array}
$$

$$
\begin{array}{lllll}
\nu & = + \quad 28''.3 & & + \quad 0''.0511\, nt & \\
 & - 206''.9 & \cos g & + 40''.9 & \sin g \\
 & + \quad 0''.6087\, nt & \cos g & - \quad 1''.6188\, nt & \sin g \\
 & - \quad 8''.2 & \cos 2g & + \quad 1''.3 & \sin 2g \\
 & + \quad 0''.0244\, nt & \cos 2g & - \quad 0''.0649\, nt & \sin 2g \\
 & \pm \text{ etc.} & & \pm \text{ etc.} &
\end{array}
$$

$$
\begin{array}{lllll}
\frac{u}{\cos i} & = + \quad 0''.4 & & + \quad 0''.3623\, nt & \\
 & - \quad 44''.2 & \sin g & - \quad 0''.7 & \cos g \\
 & - \quad 1''.3464\, nt & \sin g & - \quad 3''.0038\, nt & \cos g \\
 & - \quad 1''.5 & \sin 2g & - \quad 0''.2 & \cos 2g \\
 & - \quad 0''.0539\, nt & \sin 2g & - \quad 0''.1204\, nt & \cos 2g
\end{array}
$$

From the expressions of the perturbations that have been given, and the elements used in computing the perturbations, except that we use C in place of g_0 and the new value of the mean motion, we will compute a position of the body for the date 1894, Sept. 19, $10^h\ 48^m\ 52^s$, for which we have an observed position. From a provisional ephemeris we have an approximate value of the distance; its logarithm is 0.14878.

Reducing the above date to Berlin Mean Time, and applying the aberration time, we have, for the observed date, 1894, Sept. 19, 72800,

$$g = 339° \; 19' \; 38''.1, \qquad g' = 65° \; 24'.1.$$

Forming the arguments of the perturbations with these, we find

$$n\delta z = + \; 4' \; 43''.2, \qquad \nu = + \; 3''.6, \qquad \frac{u}{\cos i} = - \; 2''.8.$$

To convert ν into radius as unity and in parts of the logarithm of the radius vector we multiply by the modulus whose logarithm is 9.63778, and divide by 206264''.8. Thus we have from $\nu = + \; 3''.6$, the correction, + .000008, to be applied to the logarithm of the radius vector.

In case of $\frac{u}{\cos i} = - \; 2''.8$, we have

$$\delta z' = -2.''8 \; \times a \cos i = -7''.19.$$

Converting into radius as unity, we have $\delta z' = - .000035$. The coördinate z' is perpendicular to the plane of the orbit. As we will use coördinates referred to the equator we have, to find the changes in x, y, z, due to a variation of z', which we have designated by $\delta z'$, the following expressions:

$$\delta x = (\sin i \sin \Omega) \; \delta z'$$

$$\delta y = (-\sin i \cos \Omega \cos \varepsilon - \cos i \sin \varepsilon) \; \delta z'$$

$$\delta z = (-\sin i \cos \Omega \sin \varepsilon + \cos i \cos \varepsilon) \; \delta z'$$

where ε is the obliquity of the ecliptic.

For 1894 we find

$$\delta x = (-.0404) \; \delta z', \qquad \delta y = (-.3123) \; \delta z', \qquad \delta z = (+.9491) \; \delta z'$$

And for the date we have

$$\delta x = + .000001 \qquad \delta y = + .000011 \qquad \delta z = - .000033$$

With $i = 5° \ 44' \ 4''.6, \qquad \Omega = 203° \ 51' \ 51''.5, \qquad \varepsilon = 23° \ 27' \ 10''.8,$

we compute the auxiliary constants for the equator from the formulæ

$$\text{cotg}\, A = -\, tg\, \Omega \cos i, \qquad tg\, \text{E}_0 = \frac{tg\, i}{\cos \Omega},$$

$$\text{cotg}\, B = \frac{\cos i}{tg\, \Omega \cos E_0} \cdot \frac{\cos (E_0 + \varepsilon)}{\cos \varepsilon},$$

$$\text{cotg}\, C = \frac{\cos i}{tg\, \Omega \cos E_0} \cdot \frac{\sin (E_0 + \varepsilon)}{\sin \varepsilon},$$

$$\sin a = \frac{\cos \Omega}{\sin A}, \quad \sin b = \frac{\sin \Omega \cos \varepsilon}{\sin B}, \quad \sin C = \frac{\sin \Omega \sin \varepsilon}{\sin C}.$$

The values of $\sin a$, $\sin b$, $\sin c$ are always positive, and the angle E_0 is always less than 180°.

As a check we have

$$tg\, i = \frac{\sin b \sin c \sin (C - B)}{\sin a \cos A}$$

We find

$$A = 293° \ 45' \ 29''.3, \qquad B = 202° \ 59' \ 46''.9, \qquad C = 210° \ 45' \ 55''.0$$

$$\log \sin a = 9.999645, \qquad \log \sin b = 9.977735, \qquad \log \sin c = 9.498012$$

Applying $n\delta z = + \ 4' \ 43''.2$ to the value of g, we have

$$nz = 339° \ 24' \ 21''.5$$

By means of g or $nz = \text{E} - e \sin \text{E}$ we find

$$\text{E} = 337° \ 39' \ 23''.4$$

Then from

$$\sqrt{r_1} \sin \tfrac{1}{2} v = \sqrt{a(1+e)} \sin \tfrac{1}{2} \text{E}$$

$$\sqrt{r_1} \cos \tfrac{1}{2} v = \sqrt{a(1-e)} \cos \tfrac{1}{2} \text{E}$$

we find

$$v = 335° \; 50' \; 12''.2, \qquad \log r_1 = 0.378246$$

where v is the true anomaly.

Calling u the argument of the latitude we have

$$u = v + \pi - \Omega = 143° \; 52' \; 41''.8.$$

Hence

$$A + u = 77° \; 38' \; 11''.1, \qquad B + u = 346° \; 52' \; 28''.7, \qquad C + u = 354° \; 38' \; 36''.8.$$

And from

$$x = r \sin a \sin (A + u)$$

$$y = r \sin b \sin (B + u)$$

$$z = r \sin c \sin (C + u),$$

where

$$\log r = \log r_1 + \delta \log r = \log r_1 + .000008,$$

we have

$$x = + \, 2.331894, \qquad y = - \, .515433, \qquad z = - \, .070208.$$

The equatorial coördinates of the Sun for the date of the observation are

$$X = - \, 1.002563 \qquad Y = + \, .045198 \qquad Z = + \, .019611.$$

Applying the corrections δx, δy, δz, we have

$$x + \delta x + X = + \, 1.329332, \quad y + \delta y + Y = - \, .470224, \quad z + \delta z + Z = - \, .050630.$$

Then from

$$tg\ \alpha = \frac{y + \delta y + Y}{x + \delta x + X},\quad tg\ \delta = \frac{z + \delta z + Z}{y + \delta z + Y} \sin \alpha = \frac{z + \delta z + Z}{x + \delta x + X} \cos \alpha,$$

$$\Delta = \frac{z + \delta z + Z}{\sin \delta},$$

we have, giving also the observed place for the purpose of comparison,

$$\alpha_c = 340^\circ\ 31'\ 11''.4 \qquad \delta_c = -\ 2^\circ\ 3'\ 23''.1 \qquad \log \Delta = 0.149514.$$

$$\alpha_o = 340\ \ 33\ \ 49.1 \qquad \delta_o = -\ 2\ \ 2\ \ 25.4$$

where the subscript c designates the computed, and the subscript o the observed place.

Both observed and computed places are already referred to the mean equinox of 1894.0. If the observed position were the apparent place we should have to reduce the computed also to apparent place by means of the formulæ

$$\Delta\alpha = f + g \sin (G + \alpha)\ tg\ \delta$$

$$\Delta\delta = \qquad g \cos (G + \alpha),$$

the quantities f, g, and G being taken from the ephemeris for the year and date.

If the observed position has not been corrected for parallax we refer it to the centre of the Earth by means of the formulæ

$$\Delta\alpha = -\ \frac{\pi\ \rho \cos \varphi'}{\Delta} \cdot \frac{\sin (\alpha - \theta)}{\cos \delta}$$

$$tg\gamma = \frac{tg\ \varphi'}{\cos (\alpha - \theta)}$$

$$\Delta\delta = \frac{\pi\ \rho \sin \varphi'}{\Delta} \cdot \frac{\sin (\gamma - \delta)}{\sin \gamma}$$

where

α is the right ascension, δ the declination, Δ the distance of the planet from the Earth, ϕ' the geocentric latitude of the place of observation, θ the siderial time of

observation, ρ the radius of the Earth, and π the equatorial horizontal parallax of the Sun.

For the difference between computed and observed place we have

$$C - 0 = -2' \, 37''.7 \text{ in right ascension, and } C - 0 = -57''.7 \text{ in declination.}$$

By the method just given we have found the positions of the planet for several dates and have compared with the observed places. The comparison shows outstanding differences too large to be accounted for by the effects of the perturbations yet to be determined, which are the perturbations of the second order, with respect to the mass, produced by Jupiter, and the perturbations produced by the other planets that have a sensible influence. We have therefore corrected the elements that have been used in the computations thus far made, by means of differential equations formed for this purpose, employing as the absolute terms in these equations the differences between computation and observation for the several dates. A solution of the equations has given corrections to the elements that produce quite large effects on the computed place. Thus recomputing the position of the planet for the date given above with the corrected elements we find

$$\alpha_c = 340^\circ \, 33' \, 44''.5 \;, \quad \delta_c = -2^\circ \, 2' \, 15''.6.$$

And since

$$\alpha_0 = 340^\circ \, 33' \, 49''.1 \;, \quad \delta_0 = -2^\circ \, 2' \, 25''.4$$

we have, for the difference between computed and observed place,

$$C - O = -4''.6 \text{ in right ascension, and } C - O = +9''.8 \text{ in declination.}$$